Info

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Elliptic integrals and Jacobi elliptic functions

#function/real/1

the arc length of the lemniscate

Definition. The arc length of the lemniscate

$$g_0(x) := 1 - x^4$$
$$E_0(a) := \int_{[0,a]} \frac{1}{\sqrt{g_0}} = \int_{t\in[0,a]} \frac{1}{\sqrt{1-t^4}}$$

> $$E_0(u) + E_0(v) = E_0\left(\frac{u\sqrt{1-v^4}+v\sqrt{1-u^4}}{1+(uv)^2}\right)$$

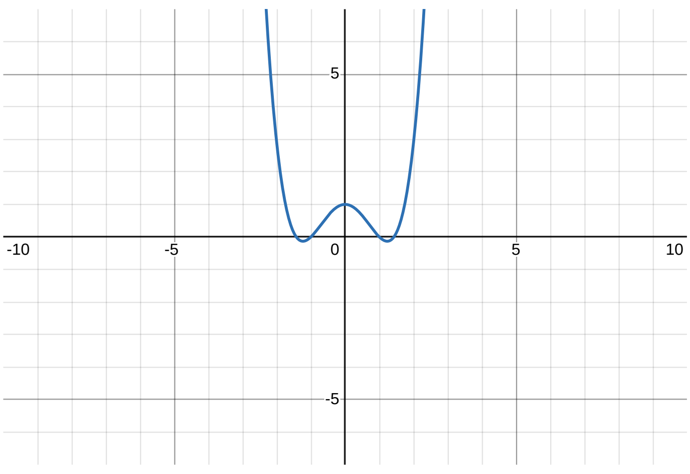
elliptic integral of the first kind

Definition. Jacobi's elliptic integral of the first kind

Let

$$g_1(x,k) := (1-x^2)(1-k^2x^2)$$

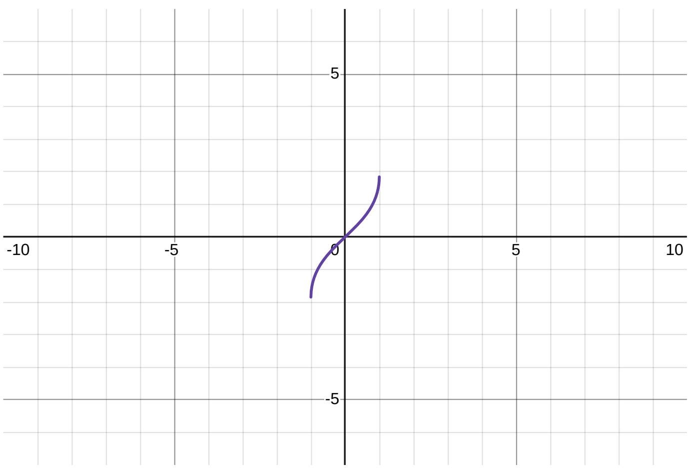
be the quartic polynomial for $k \in [0,1)$.



The **Jacobi's elliptic integral of the first kind** is the function

$$E_1(t,k) := \int_{[0,t]} \frac{1}{\sqrt{g_1(-,k)}} = \int_{x\in[0,t]} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$$
$$= \int_{\theta\in[0,\sin^{-1}(t)]} \frac{1}{\sqrt{1-k^2\sin^2\theta}}$$

where the **modulus** $0<k<1$ is fixed



and the **complete Jacobi's elliptic integral of the first kind**

$$K_1(k) := \int_{x\in[0,1]} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_{\theta\in[0,\frac{\pi}{2}]} \frac{1}{\sqrt{1-k^2\sin^2\theta}}$$

- Hence, for a fixed $k \in (0,1)$

$$E_1'(x) = \frac{1}{\sqrt{g_1(x)}}, E_1(0) = 0$$

(Euler's Addition Theorem) Let

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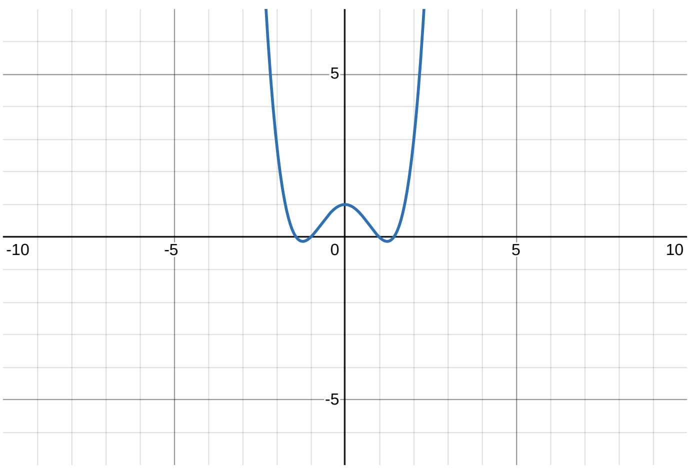

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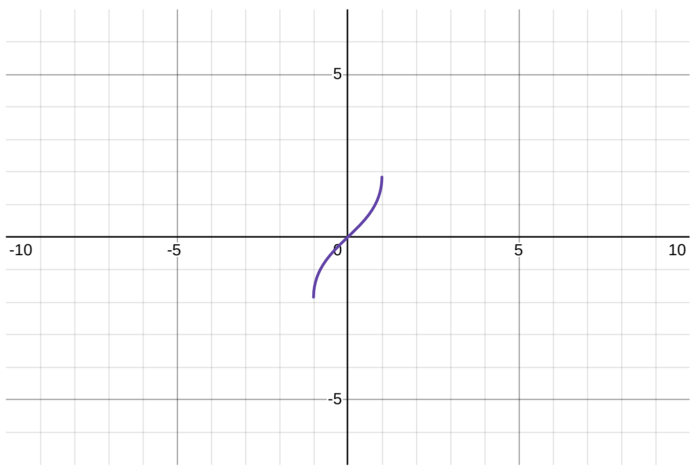

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then

$$E_1(a)+E_1(b) = E_1\left(\frac{b\sqrt{g_1(a)}+a\sqrt{g_1(b)}}{\sqrt{1-(kab)^2}}\right)$$

[1]

Definition. Coordinates on the ellipse

For $a\geq 1$ the ellipse

$$\frac{x^2}{a^2}+y^2=1$$

of eccentricity $k := \frac{\sqrt{a^2-1}}{a} = \sqrt{1-\frac{1}{a^2}} \in [0,1)$, we have two angle coordinates θ,ϕ which satisfy

$$\begin{aligned}(x,y) &= (a\cos\phi, \sin\phi)\\ \tan\theta &= \frac{y}{x}\\ r(\theta,a) &= \frac{a}{\sqrt{(\cos\theta)^2+(a\sin\theta)^2}}\\ &= \frac{1}{\sqrt{1-\frac{a^2-1}{a^2}(\sin\theta)^2}}\end{aligned}$$

and we have the *unintuitive* coordinate

$$u(\tilde{\theta}) = \int_{(0,\tilde{\theta})} r(\theta,k)\mathrm{d}\theta = E_1(\sin\tilde{\theta},k)$$

We want to define *Jacobi's elliptic functions* which produces

$$\begin{aligned}&\mathrm{sn}(u(\theta),k) &&= y(\theta) &&= \sin(\phi(\theta)) &&= r(\theta)\sin\theta\\ &\mathrm{cn}(u(\theta),k) &&= x(\theta)/a &&= \cos\phi(\theta) &&= r\cos\theta\\ &\mathrm{dn}(u,k) &&= r/a\end{aligned}$$

- Then

$$\begin{aligned}\mathrm{sn}^2+\mathrm{cn}^2 &= 1\\ \mathrm{dn}^2+k^2\mathrm{sn}^2 &= 1\end{aligned}$$

- Fix a k. The *definition*

$$u(\theta) := E_1(\sin(\theta))$$

using $\theta=\phi$ looks like

$$u(\phi) = E_1(\underbrace{\sin\phi}_{y(\theta(\phi))},k)$$

  - But again using $\theta=\phi$ we also have

$$\mathrm{sn}(u(\phi)) = y(\theta(\phi))$$

  - This gives

$$u(\phi) = E_1(\mathrm{sn}(u(\phi)))$$

  for all $\phi\in[0,2\pi]$. Thus, it must be an identity that

$$E_1(\mathrm{sn}(u)) = u$$

  making sn the *inverse* of E_1, the elliptic integral of the first kind!

Definition. Jacobi's elliptic functions

From

Definition. Jacobi's elliptic integral of the first kind

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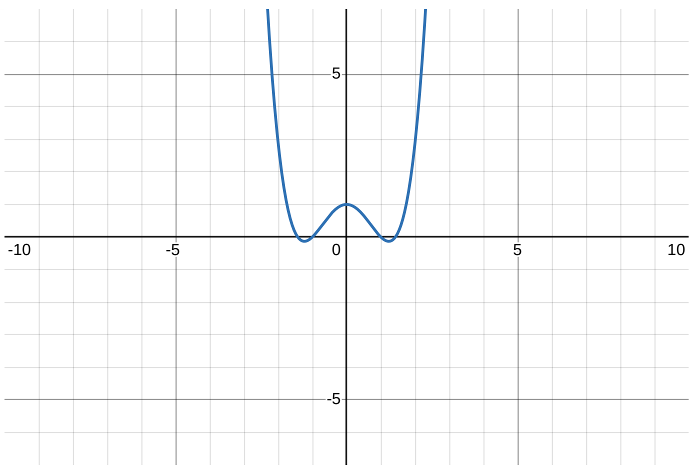

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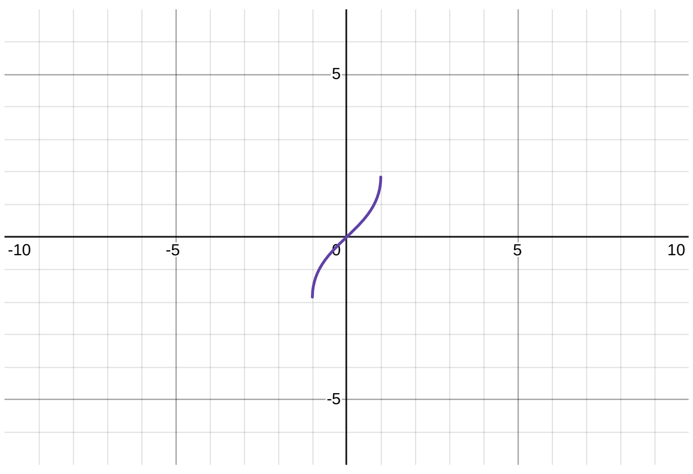

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using which we define **Jacobi's elliptic functions of elliptic modulus** k as

$$\mathrm{sn}(u) := E_1^{-1}(u)$$

- On the ellipse

$$\mathrm{d}x = -a^2\frac{y}{x}\mathrm{d}x$$
$$\mathrm{d}u = \frac{a^2}{xr}\mathrm{d}y$$

- $$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn} = \frac{xr}{a^2} = \mathrm{cn}\,\mathrm{dn}$$
$$= \sqrt{1-\mathrm{sn}^2}\sqrt{1-k^2\mathrm{sn}^2}$$

    - This is expected because

$$E_1(\mathrm{sn}(u)) = u \iff E_1'(\mathrm{sn}(u))\mathrm{sn}'(u) = 1$$
$$\iff \mathrm{sn}'(u) = \frac{1}{E_1'(\mathrm{sn}(u))}$$

    where $E_1'(x) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}$.
    - This implies

$$(\mathrm{sn}(t), \mathrm{sn}'(t)) \in \{y^2 = (1-x^2)(1-k^2x^2)\}$$

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second kind

Current note has 0 direct children and 0 total descendants.

- squishy
    - pow
        - Elliptic integrals and Jacobi elliptic functions

And it has 28 siblings.

- squishy
    - pow
        - Angle function
        - Bessel functions
        - $$\cos\sqrt{z}$$
        - The complex logarithm
        - $\sum_{n\geq 0} z^{n!}$
        - z^3-3z
        - $$\frac{r_0}{1+\epsilon\cos b\theta}$$
        - $\cos\left(\pi\frac{p}{q}\right)$
        - $\frac{\cos x^k}{x}$
        - Elliptic integrals and Jacobi elliptic functions
        - The complex exponential $\exp(z)$
        - $\exp\left(\frac{z+1}{z-1}\right)$
        - e^{-x^2}
        - [Only sum terms that are multiples of N in $\mathbb{C}[[z]](https://rupadarshiray.github.io/notes/YchnxQ9SF1RsuKjBntMv8a.$
        ]]

- $\int \frac{1}{x^2+bx+c}$
- $\frac{\exp(z)}{1-z}$
- $\sin x$
- $\sin(x \sin(x))$
- $\frac{\sin x}{x}$
- stock.sin x by xk
- $\sin x^2$
- $\sqrt{z}$
- $\sqrt{z^2-1}$
- Theta functions
- Weierstrass elliptic function $\wp$
- Weierstrass' primary factors
- $x^2 \sin \frac{1}{x}$
- $x + x^2 \sin \frac{1}{x}$

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1. scholar.rose-hulman.edu/cgi/viewcontent.cgi?article=1148&context=rhumj ↩