Info

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Vector bundles on complex 1-manifolds

Line bundles

On a Riemann surface M

$$\{\text{hol line bundles on } M\}_{\cong} \cong H^1(M, \mathscr{O}^*)$$

Let M be a Riemann surface. Then

$$p > 2 \implies H^p(M, \mathscr{O}(L)) = 0 = H^p(M, \mathbb{C}) = H^p(M, \mathbb{Z})$$

for a line bundle L.

(Serre duality) If L is a line bundle on a compact Riemann surface M then

$$H^1(M, L) \cong H^0(M, K_M \otimes L^*)^*$$

$$H^0(M, LL_p^{-1}) \cong H^0(M, L)$$

Definition. Degree of a line bundle on a Riemann surface

From the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{\exp(2\pi i -)} \mathscr{O}^* \to 1$$

which on a **compact** Riemann surface gives a long exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O}) \to H^1(M, \mathscr{O}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathscr{O}) \to \dots$$

where we have

- $\mathbb{C} \to \mathbb{C}^*$ is surjective which means $H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O})$ is injective
- $H^2(M, \mathscr{O}) = 0$
- $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$

so the equation becomes

$$0 \to \underbrace{\frac{H^1(M, \mathscr{O})}{H^1(M, \mathbb{Z})}}_{\cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong (S^1)^{2g}} \to \underbrace{H^1(M, \mathscr{O}^*)}_{\text{space of line bundles on } M} \xrightarrow{\delta} \underbrace{H^2(M, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0$$

where $\delta([L])$ is called the **degree or the first Chern class** of the line bundle L

Line bundles	**transition functions live in** $H^1(M, \mathscr{O}^*)$	**sections**	**space of sections** $H^0(M, L)$	degree
$L \to M$ with $U_\alpha \subseteq M$	$g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ (say)	s (say)		
$L_1 \otimes L_2 := L_1 L_2$	$g_{\alpha\beta}^{(L_1)} g_{\alpha\beta}^{(L_2)}$	$s_1 s_2$?		$\deg L_1 + \deg L_2$
the bundle $L \otimes L^*$ is trivial	?	holomorphic line bundle hom		0
$L^{-1} := L^*$ the dual bundle is a "multiplicative inverse"	$(g_{\alpha\beta})^{-1}$			$-\deg(L)$
the canonical bundle $K_M := T^{(1,0)} M$			holomorphic differential forms	

Line bundles	transition functions live in $H^1(M, \mathscr{O}^*)$	sections	space of sections $H^0(M, L)$	degree

L_p for any $p \in M$	On U_p with coordinate z such that $z(p) = 0$ then $g_{01} := z : U_p \cap \{p$	Has a **canonical section** s_p: take the two functions z on U_p and 1 on $\{p\}^\complement$ which becomes a section since $z = z1 = g_{01}1$ This section vanishes only at p	**Riemann surfaces**
			Definition. C co ℂ m o d 1 ca **R** **su**
			We have
			- stem.R2.C.classif
			- sett.Man.C1.T C complex tangent space
			- stem.R2.C.Hol holomorphic functions
			  - Non-constant holomorphic maps between Riemann surfaces locally looks like w^k and are open maps
			  - stem.R2.C.Hol.global
			- stem.R2.C.TC.forms differential forms
			  - Stroke's theorem
			  - de Rham
			  - ...
			- stem.R2.C.Mer meromorphic functions
			- stem.R2.C.Bun Vec
			  - line bundles
				1

$> L$ has a section that vanishes in $(p_i)_{i=1}^N$ with multiplicity m_i then it is $\bigotimes_i (L_{p_i})^{\otimes m_i}$				$\sum_i m_i$
pullback bundle $f^*L := \{(x, q) \in$	$\gamma_{\alpha\beta} \circ f$	$s : \tilde{M} \to L$ such that $\pi \circ s = f$		

Line bundles	transition functions live in $$H^1(M, \mathscr{O}^*)$$	sections	space of sections $$H^0(M, L)$$	degree
On $\mathbb{C}P^1$ we define the bundle $$\mathcal{O}(n)$$	on usual patches U_0, U_1 we have the transition function $$g_{01} := z^n : U_0 \cap U$$	a section of this line bundle is given by holomorphic $s_0, s_1 : \mathbb{C} \to \mathbb{C}$ such that $$s_0(z) = z^n s_1 \left(\frac{1}{z}\right)$$ expanding, we get $s_0 = \sum_{i=0}^n a_i z^i$	Hence the sections are just **polynomials** $$H^0(\mathbb{C}P^1, \mathcal{O}(n))$$ and thus $$\dim H^0(\mathbb{C}P^1, \mathcal{O}($$	The exact sequence becomes $$0 \to 0 \to H^1(\mathbb{C}P$$ so integers (degree) classify line bundles on $\mathbb{C}P^1$ so $\mathcal{O}(1) \cong L_p$ for some $p \in \mathbb{C}P^1$ which is of degree 1 and $$L_{p_1} L_{p_2} \dots L_{p_m} \cong$$

If $\deg L < 0$ then L only has trivial sections.

If M is a compact Riemann surface of genus g then

$$\dim H^1(M, K_M) = 2g$$

Jacobian

Riemann surfaces

Definition. Connected complex $\mathbb{C}$-manifolds of $\mathbb{C}$-dimension 1 are called **Riemann surfaces**.

We have

- stem.R2.C.classif
- sett.Man.C 1.T C complex tangent space
- stem.R2.C.Hol holomorphic functions
  - Non-constant holomorphic maps between Riemann surfaces locally looks like w^k and are open maps
  - stem.R2.C.Hol.global
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  - ...
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- stem.R2.C.Bun Vec
  - line bundles

$$\mathrm{Jac}(M) := \frac{H^1(M, \mathscr{O})}{H^1(M, \mathbb{Z})}$$
$$\mathrm{Jac}^d(M) := \{\text{line bundles of degree } d\}_{\cong} \cong \mathrm{Jac}(M)$$

There is a holomorphic map

$$M^{g-1} \to J^{g-1}$$
$$(p_1, \dots, p_{g-1}) \mapsto L_{p_1} \dots L_{p_{g-1}}$$

whose image is called the **theta divisor**.

Vector bundles

	transition functions	rank	degree
E	$g_{\alpha\beta}$		$\deg(E) := \deg(\det(E)$
$E_1 \oplus E_2$		$\mathrm{rk}(E_1) + \mathrm{rk}(E_2)$	
$E_1 \otimes E_2$			
$\det E := \Lambda^{\mathrm{rk}(E)} E$	$\det(g_{\alpha\beta})$	1	

(Riemann-Roch) If $E \to M$ is a holomorhic vector bundle on a compact Riemann surface of genus g then

$$\dim H^0(M, \mathscr{O}(E)) - \dim H^1(M, \mathscr{O}(E)) = \deg(E) + (1-g)\mathrm{rk}E$$

- If E is the trivial line bundle then $\mathscr{O}(E) = \mathscr{O}$ and
  $$\dim H^0(M, \mathscr{O}(E)) - \dim H^1(M, \mathscr{O}(E)) = 1 - g$$
  so the formula holds.
- Let L be a line bundle on M and we consider the short exact sequence
  $$0 \to \mathscr{O}(L) \xrightarrow{s_p} \mathscr{O}(LL_p) \to \mathscr{O}_p(LL_p) \to 0$$
  which gives the long exact sequence
  $$0 \to H^0(M, \mathscr{O}(L)) \to H^0(M, LL_p) \to \mathbb{C} \to H^1(M, \mathscr{O}(L)) \to H^1(M, \mathscr{O}(LL_p)) \to 0$$
  - By exactness the alternating sum of the dimensions must be zero therefore

(Birkoff-Grothendieck) If E is a rank m holomorphic vector bundle over $\mathbb{C}P^1$ then

$$E \cong \bigoplus_{i=1}^{m} \mathscr{O}(a_i)$$

for some $a_i \in \mathbb{Z}$.

- > Let E be a holomorphic vector bundle over $\mathbb{C}P^1$ then E is trivial $\iff$
  > $$\deg E = 0 \text{ and } H^0(\mathbb{C}P^1, E(-1)) = 0$$

$$\deg f := \deg(f^* L_p)$$

Let $f : N \to M$ be a holomorphic map between compact Riemann surfaces and L is a line bundle on N. Then

$$H^0(M, f_* \mathscr{O}(L)) \cong H^0(N, \mathscr{O}(L))$$

Any rank $= \deg f$ holomorphic vector bundle $E \to M$ has

$$f_* \mathscr{O}(L) = \mathscr{O}(E)$$

If $V \to M$ is a holomorphic vector bundle then

$$f_* \mathscr{O}(L \otimes f^* V) \cong \mathscr{O}(E \otimes V)$$

> Each point $p \in M$ has a nbd U for which
> $$f_* \mathscr{O}(L)(U) \cong \bigoplus_{i=1}^{m} \mathscr{O}(U)$$

> With L, E as above
> $$\deg E = \deg L + (1 - g(N)) - (1 - g(M)) \deg f$$

$$f : N \to \mathbb{C}P^1$$

then

$$\deg E = \deg L + (1 - g(N)) - \deg f$$

On $\mathbb{C}P^1$

For

$$f : \underbrace{M}_{\text{genus } g} \to \mathbb{C}P^1$$

for a generic line bundle $L(-1) \in J^{g-1}$ will give

$$f_* \mathscr{O}(L) \to \mathbb{C}P^1$$

will be the trivial bundle E over $\mathbb{C}P^1$.

We now take a section

$$w \in H^0(M, f^* \mathcal{O}(n))$$

Let $U \subseteq \mathbb{C}P^1$ then multiplication by w defines a linear map

$$w(-) : H^0(f^{-1}(U), L) \to H^0(f^{-1}(U), L(n))$$

By definition of E this is a homomorphism

$$W : H^0(U, E) \to H^0(U, E(n))$$

Since W is globally defined and E is trivial

$$W : H^0(\mathbb{C}P^1, E) \to H^0(\mathbb{C}P^1, E(n)) \cong \mathbb{C}^n \otimes H^0(\mathbb{C}P^1, \mathcal{O}(n))$$

which is a $m \times m$ matrix-valued holomorphic section of $\mathcal{O}(n)$ which is a polynomial degree $\leq n$ with matrix coefficients

$$A(z) = A_0 + A_1 z + \cdots + A_n z^n$$

- Moduli space of $\mathbb{C}$-manifolds
- Meromorphic functions on a Riemann surface
- Normalization of plane projective $\mathbb{C}$-curves and Riemann surfaces
- Riemann surfaces defined over a number field
- Hyperelliptic Riemann surfaces
- Riemann surfaces with a holomorphic 1-form