Info

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# Manifold of Gibbs distribution

**Definition.** Gibbsian manifold of probability measures

- We start with

$$(X, \mu, A)$$

a measure space $(X, \mu)$ and some measurable functions $A : X \to \mathbb{R}^n$
- Let $\Lambda \subseteq \mathbb{R}^n$ be the largest open set such that

$$Z : \Lambda \to \mathbb{R}$$
$$\lambda \mapsto \int_X \exp\left(-\sum_{i=1}^n \lambda_i A_i(x)\right)$$

exists.
- We have the *exponential* probability measures

$$\pi : \Lambda \to \mathbb{P}(X)$$
$$\lambda \mapsto \pi_\lambda(x) := \frac{\exp\left(-\sum_{i=1}^n \lambda_i A_i(x)\right)}{Z(\lambda)}$$

- $$Q := \{x \in \mathbb{R}^n : \langle A_i \rangle = x_i\}$$
- Notice

$$-\frac{\partial}{\partial \lambda_i} \log Z(\lambda) = \langle A_i \rangle_{\pi_\lambda}$$

which means $(\partial_{\lambda_i}(-\log Z)) : \Lambda \to Q$ and because its jacobian is the *covariant matrix of* $A_i$ its determinant

$$\det[\partial_{\lambda_j}\partial_{\lambda_i} - \log Z] \neq 0$$

> - So, $(\partial_{\lambda_i}(-\log Z)) : \Lambda \to Q$ is a local diffeomorphism [1], now we assume the inverse function
>
> $$p : Q \to \Lambda$$
>
> (that is, such that $\partial_{\lambda_i} \log Z(p(q)) = q_i$) exists.

$$S(\pi_{\lambda = p(q)}) \geq S(\rho)$$

for any $\rho \in \{\rho \in \mathrm{L}^1(X, \mu) : \rho \geq 0, \int_X \rho = 1, \int_X A_i \rho = q_i\}$

- On $Q$, then we have maps

$$\pi : Q \to \mathbb{P}(X)$$
$$q \mapsto \pi_q(x) := \frac{\exp\left(-\sum_{i=1}^n p_i(q) A_i(x)\right)}{Z(p(q))}$$

such that $\langle A_i \rangle_q = q_i$, hence giving us a Gibbsian manifold of measures $(Q, \pi)$.
- Now, assuming(?) $S : Q \to \mathbb{R}$ is smooth, we have a configuration manifold with a smooth map where

$$\mathrm{d}S = p_i(q)\mathrm{d}q_i$$

Thus we have went through

$$(X, \mu, A) \to (Q, p, \backslash p[](sett.cnstr.Man$$

whose image is defined as $Q$. $\Lambda$ be the largest open set such that

$$Z : \Lambda \subseteq V \to \mathbb{R}$$
$$Z(\lambda) := \int_X \exp(-A^\lambda(x)$$

converges, and observe

$$\mathrm{d}(-\log Z) : \Lambda \subseteq V \to Q \subseteq V^*$$
$$\lambda \mapsto \langle A \rangle_{\pi_\lambda}$$

is locally invertible, so we pick an inverse

$$\beta : Q \to \Lambda$$

which is a 1-form on $Q$ and define

$$\pi_{\beta(-)} : Q \to \mathbb{P}(X)$$

whose entropy

$$S : Q \to \mathbb{R}$$

has derivative

$$\mathrm{d}S = \beta$$

This captures canonical Gibbs distributions, Lie group thermodynamics,

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- with $(Q, S)$

> - Is it a system of convex space with a concave map?
> - **Symplectic geometry**: We may obtain the Lagrangian submanifold of $T^*Q$
> - **Contact geometry**: $(\mathbb{R} \times T^*Q, \mathrm{d}S - p_i \mathrm{d}q_i)$?
> - Thermodynamic functions; Legendre transforms

- with $(Q, \pi)$

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1. for $n = 1$

$$\partial^2_{\lambda_i} \log Z \geq 0$$

we have global inverse ↩

2. Classical Mechanics (ucr.edu) ↩
3. Extremal Principles in Classical, Statistical and Quantum Mechanics | Azimuth (wordpress.com) ↩