Info

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Intersection of open, dense subsets of a topological space

Definition. Meager subsets

Let X be a topological space. A subset $S \subseteq X$ is

- **meager** if S is the union of countably many closed sets with empty interior
- **comeager** if S is the intersection of countably many open, dense sets

A topological space X is called **Baire** if every comeager subset is open and dense.

Consider a meager set

$$\bigcup_{n\geq 1} \underbrace{K_n}_{\substack{\text{closed} \\ \text{int}=\emptyset}}$$

Its completement

$$\bigcap_{n\geq 1} \underbrace{K_n^{\mathsf{c}}}_{\substack{\text{open,} \\ \text{dense}}}$$

is comeager.

for complete metric spaces

(complete metric spaces are Baire) Let $\{S_k\}_{k\geq 1}$ be a sequence of open, dense subsets of a nonempty complete metric space X. Then the intersection

$$\bigcap_{k\geq 1} S_k$$

is a nonempty, dense subset of X.

> **Proposition:** Let $\Omega \subseteq X$ be any open set. We have $\left(\bigcap_{k\geq 1} S_k\right) \cap \Omega$ is nonempty.

- **Base case:** Let $x_0 \in X$ and $r_0 < 1$ such that $B(x_0, 3r_0) \subseteq \Omega$.
- **Induction hypothesis:** Assume for $k \geq 1$, there exists x_k, r_k such that

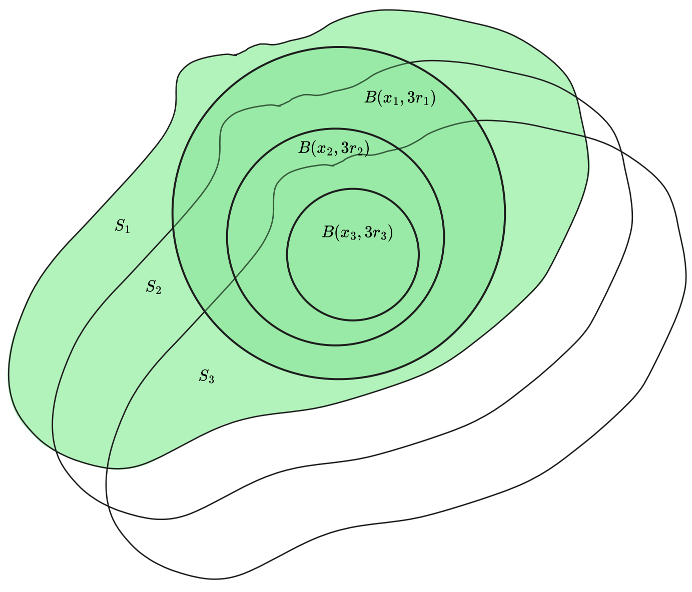
$$B(x_k, 3r_k) \subset S_k \cap B(x_{k-1}, r_{k-1})$$

- **Inductive step:** Consider

$$\underbrace{S_{k+1}}_{\substack{\text{open} \\ \text{dense}}} \cap \underbrace{B(x_k, r_k)}_{\text{open}}$$

  - Because S_{k+1} is *dense*, the intersection is non-empty.
  - Because both sets are *open*, the intersection is open.
  - Therefore, there exists x_{k+1}, r_{k+1} such that

$$B(x_{k+1}, 3r_{k+1}) \subset S_{k+1} \cap B(x_k, r_k)$$



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$$\begin{aligned} x_{k+1} &\in B(x_k, r_k) \\ \implies d(x_{k+1}, x_k) &\leq r_k \end{aligned}$$

- However,

$$r_{k+1} \leq \frac{r_k}{3}$$

- This implies

$$\begin{aligned} d(x_{k+1}, x_k) &\leq r_k \\ &\leq \frac{r_{k-1}}{3} \\ &\leq \frac{r_{k-2}}{3^2} \\ &\leq \dots \\ &\leq \frac{r_0}{3^k} \\ &\leq \frac{1}{3^k} \end{aligned}$$

- Thus, $\{x_k\}_{k\geq 1}$ is Cauchy and by completeness of X we have a limit

$$x_k \xrightarrow{k\to\infty} x_\infty$$

for some $x_\infty \in X$.
- We now observe that

$$\begin{aligned} d(x_\infty, x_k) &\leq \sum_{j=k}^{\infty} d(x_{j+1}, x_j) \\ &\leq 3^k r_k \sum_{j=k}^{\infty} \frac{1}{3^j} \\ &\leq \frac{3}{2} r_k \end{aligned}$$

- Therefore, for every $k \geq 1$ we have

$$x_\infty \in B(x_k, 3r_k) \subseteq S_k$$

and

$$x_\infty \in B(x_1, 3r_1) \subseteq B(x_0, 3r_0) \subseteq \Omega$$

- Therefore,

$$x_\infty \in \left(\bigcap_{k\geq 1} S_k\right) \cap \Omega$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Intersection of open, dense subsets of a topological space

And it has 29 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
    - Discrete subgroup acting on coset space
    - Basis and subbasis for a topology
    - Principal G-bundles over topological spaces

$$G \curvearrowright P \to X$$

    - Vector bundles over topological spaces
    - Closure of a subset of topological space
    - Locally compact and 2-countable Hausdorff topological spaces are paracompact with countable refinement
    - Covering dimension of topological spaces
    - $\mathrm{EndTop}(X, Y)$
    - Eilenberg–Steenrod functors

$$(H_n : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Ab}, \partial_n)$$

    - Local fields
    - Local fields of characteristic 0
    - Topological groups
    - Continuous group action on a topological space
    - Intersection of open, dense subsets of a topological space
    - Local topological spaces
    - Covering space of a topological space
    - Borel measures on a topological space
    - Convergence for nets in a topological space
    - Loops and a open cover of a topological space
    - Paths in a topological space
    - Product of topological spaces
    - Ringed topological spaces
    - Sheaf on open sets of a topological space
    - Closed subsets of a topological space
    - Discrete subsets of topological spaces
    - Types of topological spaces
    - Locally compact Hausdorff spaces
    - Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$