Info

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T-closed, closed subgroups of $GL(n, \mathbb{R})$

Let G be a T-closed, closed subgroups of $GL(n, \mathbb{R})$.

Definition. T-closed subgroups of $GL(n, \mathbb{R})$

A closed subgroup $G \leq GL(n, \mathbb{R})$ is **T-closed** if it is stable under the transposition of matrices

$$A \in G \implies A^\intercal \in G$$

- T-closed subgroups $G \leq GL(n, \mathbb{R})$ have the order 2 automorphism

$$\begin{aligned} \Theta : G &\to G \\ g &\mapsto (g^{-1})^\intercal \end{aligned}$$

Definition.

$$K := G \cap O(n, \mathbb{R}) = \{g \in G | \Theta(g) = g\}$$

- Its derivative is an order 2 automorphism

$$\begin{aligned} \theta : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto -X^\intercal \end{aligned}$$

- Thus, the Lie algebra of such a T-closed group is

  Definition. An **involutive Lie algebra** $(\mathfrak{g}, s)$ is a Lie $\mathbb{R}$-algebra $\mathfrak{g}$ with an automorphism $s \in \mathrm{AutLieAlg}_{\mathbb{R}}(\mathfrak{g})$ of order 2.

- Consider the inner product

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ B(X, Y) &= \mathrm{tr}(XY^\intercal) \end{aligned}$$

  - Then

$$\begin{aligned} B(\theta(X), \theta(Y)) &= \mathrm{tr}((-X^\intercal)(-Y)) \\ &= \mathrm{tr}(X^\intercal Y) \\ &= \mathrm{tr}(Y^\intercal X)^\intercal \\ &= B(Y, X) \\ &= B(X, Y) \end{aligned}$$

By

Endomorphisms of FinVec_k which are of order 2 are diagonalizable

Let $A \in \mathrm{End}\mathsf{FinVec}_k(V)$ with $A^2 = \mathsf{Id}$. Then

- either $A = \mathsf{Id}$
  - thus A is already diagonal
- or $A \neq \mathsf{Id}$
  - then characteristic polynomial is

$$X^2 - 1 = (X - 1)(X + 1)$$

    and minimal polynomial must be a factor, so it is linear
  - $\implies$ A is diagonalizable

we may decompose

Definition.

$$\mathfrak{g} = \underbrace{\mathfrak{K}}_{\ker(\theta - \mathsf{Id})} + \underbrace{\mathfrak{p}}_{\ker(\theta + \mathsf{Id})}$$

where

$$\begin{aligned} \mathfrak{K} &= \mathfrak{g} \cap \mathfrak{so}(n, \mathbb{R}) \\ \mathfrak{p} &= \mathfrak{g} \cap \mathsf{Sym}(n, \mathbb{R}) \end{aligned}$$

Proposition:

- $$\begin{aligned} [\mathfrak{K}, \mathfrak{K}] &\leq \mathfrak{K} \\ [\mathfrak{K}, \mathfrak{p}] &\subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] &\subseteq \mathfrak{K} \end{aligned}$$

- $$\mathfrak{K} = T_I K$$

  where

  Definition.

$$K := G \cap O(n, \mathbb{R}) = \{g \in G | \Theta(g) = g\}$$

$$\begin{aligned} \theta[X, Y] &= [\theta(X), \theta(Y)] \\ X \in \mathfrak{K}, Y \in \mathfrak{p} \implies \theta[X, Y] &= -[X, Y] \end{aligned}$$

B-adjoint of ad_H

Let $H = H^\top$, that is, $H \in \mathfrak{p}$

$$\begin{aligned} B(\mathrm{ad}_H(X), Y) &= B([H, X], Y) \\ &= \mathrm{tr}((HX - XH)Y^\top) \\ &= \mathrm{tr}(HXY^\top - XHY^\top) \\ &= \mathrm{tr}(XY^\top H - XHY^\top) \\ &= \mathrm{tr}(XY^\top H^\top - XH^\top Y^\top) \\ &= \mathrm{tr}(X(HY - YH)^\top) \\ &= B(X, \mathrm{ad}_H(Y)) \end{aligned}$$

But in general

$$\begin{aligned} B(\mathrm{ad}_H(X), Y) &= B([H, X], Y) \\ &= \mathrm{tr}((HX - XH)Y^\top) \\ &= \mathrm{tr}(HXY^\top - XHY^\top) \\ &= \mathrm{tr}(XY^\top H - XHY^\top) \\ B(X, -\mathrm{ad}_{\theta(H)}(Y)) &= B(X, [-\theta(H), Y]) \\ &= -\mathrm{tr}(X(-H^\top Y + YH^\top)^\top) \\ &= \mathrm{tr}(XY^\top H - XHY^\top) \end{aligned}$$

Thus

> **Proposition:**
>
> $$B(\mathrm{ad}_H(X), Y) = B(X, -\mathrm{ad}_{\theta(H)}(Y))$$
>
> and in particular if $H \in \mathfrak{p}$, then ad_H is B-symmetric.

restricted root space decomposition

- As $\dim \mathfrak{p} < \infty$ and one dimensional subspaces are Abelian, there exists maximal Abelian subspaces of $\mathfrak{p}$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal Abelian subspace.
  - Then

    $$\mathrm{ad}(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{g})$$

    is a commuting family of B-symmetric linear maps by

    > **Proposition:**
    >
    > $$B(\mathrm{ad}_H(X), Y) = B(X, -\mathrm{ad}_{\theta(H)}(Y))$$
    >
    > and in particular if $H \in \mathfrak{p}$, then ad_H is B-symmetric.
  - ...
  - Thus, it admits an orthogonal diagonalization

    $$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \mathfrak{a}^*} \mathfrak{g}_\alpha$$

    called **restricted root space decomposition** of $(\mathfrak{g}, \mathfrak{a})$ where $\alpha \in \mathfrak{a}^* \setminus \{0\}$ such that

    $$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} | \mathrm{ad}(H)X = \alpha(H)X, \forall H \in \mathfrak{a}\} \neq 0$$

    is called a **restricted root space** of $(\mathfrak{g}, \mathfrak{a})$ and α its **restricted root**

    $$\Sigma := \{\alpha \in \mathfrak{a}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq 0\}$$

    .

> **Proposition:**
>
> - $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
> - $\theta(\mathfrak{g}_\alpha) = \theta(\mathfrak{g}_{-\alpha})$
> - $\mathfrak{g}_0 = \mathfrak{a} + Z_{\mathfrak{K}}(\mathfrak{a})$
> - Σ is finite

?

$$[H, \theta(X)] = \theta[\theta(H), X] = -\theta([H, X]) = -\alpha(H)\theta(X)$$

We have $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$, hence

$$\mathfrak{g}_0 = \mathfrak{K} \cap \mathfrak{g}_0 + \mathfrak{p} \cap \mathfrak{g}_0$$

- Here, $\mathfrak{p} \cap \mathfrak{g}_0$ is Abelian because

  $$H \in \mathfrak{p} \cap \mathfrak{g}_0 \implies [H, Y] = \underbrace{\alpha(H)}_{0} Y = 0$$

- Since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g}_0$ and it is maximal Abelian in $\mathfrak{p}$ we have

  $$\mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$$

- Now

  $$\begin{aligned} X \in \mathfrak{K} \cap \mathfrak{g}_0 &\iff \mathrm{ad}_X = 0, X \in \mathfrak{K} \\ &\iff X \in Z_{\mathfrak{K}}(\mathfrak{a}) \end{aligned}$$

> Definition.
>
> $$Z_{\mathfrak{b}}(\mathfrak{c}) = \{s \in \mathfrak{g} | \mathrm{ad}_s|_{\mathfrak{c}} = 0\} \cap \mathfrak{b}$$

[1]

Thus we have

> **Proposition:** Let $G \leq GL(n, \mathbb{R})$ be closed, T-closed. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal Abelian subspace. Then Lie algebra of G, $\mathfrak{g}$ decomposes B-orthogonally as
>
> $$\mathfrak{g} = \underbrace{\mathfrak{a}}_{\subseteq \mathfrak{p}} + \underbrace{\mathfrak{m}}_{= Z_{\mathfrak{K}}(\mathfrak{a})} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

> **Proposition:**

- As Σ is finite, $\mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker\alpha \neq \emptyset$
- Let $H \in \mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker\alpha$. Then

$$Z_{\mathfrak{g}}(H) = \mathfrak{a} + \mathfrak{m}$$

- This implies $Z_{\mathfrak{p}}(H) = Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a}$
- Thus, all maximal Abelian subspaces $\mathfrak{a} \subseteq \mathfrak{p}$ can be written as $\mathfrak{a} = Z_{\mathfrak{p}}(H)$ for some $H \in \mathfrak{a}$.

Suppose $X \in Z_{\mathfrak{g}}(H)$ Then by

Proposition: Let $G \leq GL(n, \mathbb{R})$ be closed, T-closed. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal Abelian subspace. Then Lie algebra of G, $\mathfrak{g}$ decomposes B-orthogonally as

$$\mathfrak{g} = \underbrace{\mathfrak{a}}_{\subseteq\mathfrak{p}} + \underbrace{\mathfrak{m}}_{=Z_{\mathfrak{K}}(\mathfrak{a})} + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

we have

$$X = X_0 + \sum_\alpha X_\alpha$$

B-orthogonally.

- Then

$$0 = [H, X] = \underbrace{[H, X_0]}_{=0} + \sum_\alpha \underbrace{\alpha(H)}_{\neq 0} X_\alpha$$

which implies $X_\alpha = 0$ for all $\alpha \in \Sigma$.

regular lines in $\mathfrak{p}$

Consider the function

$$\begin{aligned} \boldsymbol{P}(\mathfrak{p}) &\to \mathbb{N} \\ \mathbb{R}\langle X\rangle &\mapsto \dim Z_{\mathfrak{p}}(X) \end{aligned}$$

Definition. Let $X \in \mathfrak{p}$. Then X is **regular** if

$$\dim Z_{\mathfrak{p}}(X) \leq \dim Z_{\mathfrak{p}}(Y)$$

for all $Y \in \mathfrak{p}$.

Proposition: Let $X \in \mathfrak{p}$. Then X is contained in a **unique** maximal Abelian subspace $\mathfrak{a} \subseteq \mathfrak{p} \iff X$ is *regular*.

- ($\Longleftarrow$) If X is regular and $\mathfrak{a}$ is maximal Abelian containing X. Then

$$X \in \mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker\alpha \implies \mathfrak{a} = Z_{\mathfrak{p}}(X)$$

- ($\Longrightarrow$) ?

Ad_K acts transitively on maximal Abelian subspaces of $\mathfrak{p}$

Proposition: If $\mathfrak{a}, \mathfrak{a}' \subseteq \mathfrak{p}$ are maximal Abelian subspaces, then there is a $k \in K$ such that

$$\mathrm{Ad}_k(\mathfrak{a}') = \mathfrak{a}$$

And as every $X \in \mathfrak{p}$ is in some maximal abelian subspace, we have

$$\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}_k(\mathfrak{a})$$

By

Proposition:

- As Σ is finite, $\mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker\alpha \neq \emptyset$
- Let $H \in \mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker\alpha$. Then

$$Z_{\mathfrak{g}}(H) = \mathfrak{a} + \mathfrak{m}$$

- This implies $Z_{\mathfrak{p}}(H) = Z_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a}$
- Thus, all maximal Abelian subspaces $\mathfrak{a} \subseteq \mathfrak{p}$ can be written as $\mathfrak{a} = Z_{\mathfrak{p}}(H)$ for some $H \in \mathfrak{a}$.

we can write

$$\begin{aligned} \mathfrak{a} &= Z_{\mathfrak{p}}(H) \\ \mathfrak{a}' &= Z_{\mathfrak{p}}(H') \end{aligned}$$

- Consider

$$\begin{aligned} K &\to \mathbb{R} \\ k &\mapsto B(\mathrm{Ad}_k(H'), H) \end{aligned}$$

- By compactness of $K = G \cap O(n, \mathbb{R})$, we have a minimizer $k_0 \in K$.
- Consider $Z \in \mathfrak{K}$

$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ t &\mapsto B(\mathrm{Ad}_{\exp(rZ)}(\mathrm{Ad}_{k_0} H'), H) \end{aligned}$$

which is a smooth function which has a minimum at $r = 0 \implies \exp(rZ) = \mathrm{Id}$ whose derivative at $r = 0$ is

$$0 = B(\mathrm{ad}_Z(\mathrm{Ad}_{k_0} H'), H) = B(Z, \underbrace{[\underbrace{\mathrm{Ad}_{k_0} H'}_{\in\mathfrak{p}}, H]}_{\in\mathfrak{K}})$$

- As this true for all $Z \in \mathfrak{K}$ we have

$$\begin{aligned} [\mathrm{Ad}_{k_0} H', H] &= 0 \\ \mathrm{Ad}_{k_0}(H') &\in Z_{\mathfrak{p}}(H) = \mathfrak{a} \end{aligned}$$

- Because $\mathfrak{a}$ is Abelian

$$\begin{aligned} J \in \mathfrak{a}, \mathrm{Ad}_{k_0}(H') \in \mathfrak{a} \implies & [J, \mathrm{Ad}_{k_0}(H')] = 0 \\ \implies & \mathfrak{a} \subseteq Z_{\mathfrak{p}}(\mathrm{Ad}_{k_0}(H')) \\ & = \mathrm{Ad}_{k_0}(Z_{\mathfrak{p}}(H')) \\ & = \mathrm{Ad}_{k_0}(\mathfrak{a}') \end{aligned}$$

- By maximality of $\mathfrak{a}$ we have

$$\mathfrak{a} = \mathrm{Ad}_{k_0}(\mathfrak{a}')$$

Current note has 0 direct children and 0 total descendants.

- stem
  - $GL(n)(\mathbb{R})$
    - T-closed, closed subgroups of $GL(n, \mathbb{R})$

And it has 4 siblings.

- stem
  - $GL(n)(\mathbb{R})$
    - T-closed, closed subgroups of $GL(n, \mathbb{R})$
    - Subgroups of $GL(n, \mathbb{R})$
    - Compact subgroups of $GL(n)(\mathbb{R})$
    - T-closed, exp-closed closed subgroups of $GL(n, \mathbb{R})$ $\leftrightarrow$ completely geodesic submanifolds of Pd$(n, \mathbb{R})$

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1. Knapp - Beyond2, p.388 ↩