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Integral of holomorphic differential forms

Definition. Integration of k -forms on $\mathcal{C}^1$ -singular chains on $U \subseteq \mathbb{C}$

- Integration of a 0 -form on a 0 -chain are just sums of values
- Integration of a 1 -form $f dz + g d\bar{z}$ on a 1 -chain is defined using integration on curves and extended $\mathbb{R}$ -linearly

$$\int_{\sum_{i=1}^n \alpha_i \gamma_i} f dz + g d\bar{z} = \sum_{i=1}^n \alpha_i \int_{\gamma_i} f dz + g d\bar{z}$$

- We may integrate 2 -form on a 2 -chain B by

$$\int_B f dz \wedge d\bar{z}$$

The holomorphic differential complex looks like

$$\begin{array}{ccccc} \mathcal{C}^{\mathcal{H}}(U, \mathbb{C}) \leq \Omega^0(U, \mathbb{C}) & \rightarrow & \Omega^{1,0}(U, \mathbb{C}) & \rightarrow & 0 \\ \phi & \xmapsto{d} & d\phi = \frac{\partial \phi}{\partial z} dz + \frac{\partial \phi}{\partial \bar{z}} d\bar{z} & & \\ & & f dz & \xmapsto{d} & 0 \end{array}$$

Thus for the holomorphic case, we restrict to *holomorphic 1-forms*

$$f dz$$

where f is a holomorphic function and 2 -forms I'm calling *holomorphic vector area form*?

$$f dz \wedge d\bar{z}$$

where f is again a holomorphic function - which are *not* actually holomorphic 2 -forms, because the only holomorphic 2 -form is 0 .

Forms like

$$|f| dx \wedge dy$$

also maybe?

stem.Rf.1Hol.Hol.Cauchy

Cauchy integral formula for holomorphic functions

holomorphic

(Cauchy integral formula) Let f be holomorphic in a disk around $z \in \mathbb{C}$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{S_r^1} \frac{f(w)}{(w-z)^{n+1}} dw$$

- By the [Cauchy integral formula](#),

$$\begin{aligned} \frac{|f^{(n)}(0)|}{n!} &\leq \frac{1}{2\pi} \frac{\|f|_{rS^1}\|_{\infty}}{r^{n+1}} (2\pi r) \\ &= \frac{\|f|_{rS^1}\|_{\infty}}{r^n} \end{aligned}$$

at the point of singularity

holomorphic forms $f(z)dz$

The form

$$f(z)dz$$

is closed *if and only if* f is holomorphic.

It is exact *if and only if* its integral is 0 identically.

(Conservative iff exact) Let $U \subseteq \mathbb{C}$ be any open set and $f : U \rightarrow \mathbb{C}$ be holomorphic. Then for all piecewise $\mathcal{C}^1$ closed curves γ on U ,

$$\int_{\gamma} f(z) dz = 0$$

if and only if there exists a holomorphic function $F : U \rightarrow \mathbb{C}$ such that

$$\frac{\partial F}{\partial z} = f \text{ on } U$$

The form

$$f(z)dz$$

is exact implies its closed, implies its holomorphic.

☰ (Morera's theorem) Any continuous function $f : U \rightarrow \mathbb{C}$ on open set $U \subseteq \mathbb{C}$ that satisfies

$$\int_{\gamma} f(z) dz = 0$$

for every $\mathcal{C}^1$ closed curve γ in U must be holomorphic on U .

Of course the converse is not true for all holomorphic forms because

$$\int_{S^1} \frac{1}{z} dz = 2\pi i$$

and the general statement is actually topological: depends on the topology of the domain of f .

☰ (Cauchy-Goursat's theorem for convex open sets) If U is a *convex open set*, $f : U \rightarrow \mathbb{C}$ is holomorphic, then

$$\int_{\vec{abc}} f(z) dz = 0$$

for every $a, b, c \in U$. This means

$$\int_{\gamma} f(z) dz = 0$$

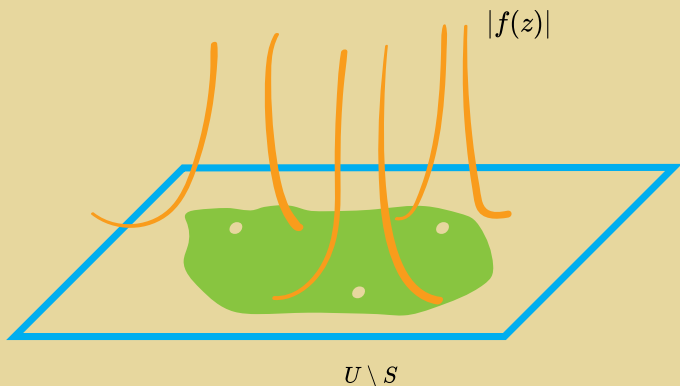
for any piecewise $\mathcal{C}^1$ closed curve γ on C .

Hence, holomorphic forms on convex sets are exact.

It is also true for simply connected sets, nothing but property of closed complex forms.

☰ (Extended Cauchy-Goursat's theorem on convex sets) Let $U \subseteq \mathbb{C}$ be a convex open set and $S \subseteq U$ be a discrete subset. If a function $f : U \rightarrow \mathbb{C}$ holomorphic on $U \setminus S$ satisfies

$$s \in S \implies \lim_{z \rightarrow s} (z - s)f(z) = 0$$



then for any curve γ in $U \setminus S$ we have

$$\int_{\gamma} f(z) dz = 0$$

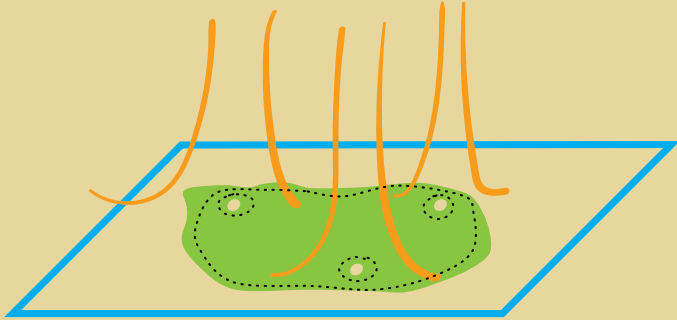
homology invariance of the integral

Because $f(z)$ is holomorphic if and only if $f(z)dz$ is closed,

$$\int_B \cancel{d(f(z)dz)}^0 = \int_{\partial B} f(z) dz$$

so its integral over homologically trivial cycles is 0 identically.

☰ (Cauchy-Goursat's theorem for any open set) If $f(z)$ is holomorphic in an open set U and γ is a cycle homologous to zero in U



then

$$\int_{\gamma} f(z) dz = 0$$

But this does not imply $f(z)$ is exact on U , obviously. Nothing "*analytic*" here, only closed complex form.

We can use that to derive at

☰ (Cauchy integral with residues formula) Given a null homologous path in open $U \subseteq \mathbb{C}$ and a holomorphic function $f : U \setminus S \rightarrow \mathbb{C}$ where $A \subset U$ is a set without limit points in U we have

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{s \in S} \text{Wind}(\gamma, s) \text{Res}(f, s)$$

where $\text{Wind}(\gamma, s) = 0$ for all but finitely many $s \in S$.

This implies if we have a meromorphic function with all residues equal to 0, then we have a anti-derivative for f in the domain of definition.

Does $\int_{\partial B} f(z) dz = 0$ for all 2-chains B mean $f(z)dz$ is closed?

homotopy invariance of the integral

weird complex forms using holomorphic functions

- Given a holomorphic 1-form $f\mathrm{d}z$ then

$$\bar{f}\mathrm{d}\bar{z}$$

is an *anti-holomorphic* 1-form and it has similar but complex conjugate properties to the holomorphic forms.

- Given a holomorphic f ,

$$f\mathrm{d}\bar{z}$$

has very different properties,

- conjugate is

$$\bar{f}\mathrm{d}z$$

- derivative is

$$d(f\mathrm{d}\bar{z}) = \mathrm{d}f \wedge \mathrm{d}\bar{z} = \frac{\partial f}{\partial z}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

hence

$$\int_B \frac{\partial f}{\partial z}\mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{\partial B} f\mathrm{d}\bar{z}$$

Does Cauchy-Goursat's statement not hold for $f(z)\mathrm{d}\bar{z}$ with holomorphic $f(z)$?

Well,

$$\int_{S^1_{(R)}} z^k \mathrm{d}\bar{z} = R^k e^{ik\theta} \mathrm{d}(Re^{-i\theta}) = (-i)R^{k+1} \int e^{i(k-1)\theta} \mathrm{d}\theta = \begin{cases} -2\pi R^2 i & k > 1 \\ 0 & k = 1 \\ 0 & k = 0 \\ 0 & k < 0 \end{cases}$$

✂ If f is holomorphic on a closed disk D of radius R then

$$\int_D \frac{\partial f}{\partial z} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{S^1_{(R)}} f \mathrm{d}\bar{z} = (-2\pi R^2 i) f'(0)$$

So

$$\int_D F \mathrm{d}x \wedge \mathrm{d}y = (2\pi R^2) F(0)$$

which is the harmonic function average property.

For a injective f holomorphic on open unit disk

$$\int_{\mathbb{D}} \left| \sum_{n \geq 0} b_n z^n \right|^2 \mathrm{d}x \wedge \mathrm{d}y = \pi \sum_{n \geq 0} |b_n|^2$$

Current note has 0 direct children and 0 total descendants.

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And it has 4 siblings.

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