

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on March 17, 2024 10:23:35 AM, was last modified on April 27, 2024 8:37:26 AM, and was exported on September 7, 2026 3:10:00 PM.

Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

any holomorphic function can be uniformly approximated by rational functions on a compact set with poles outside that compact set

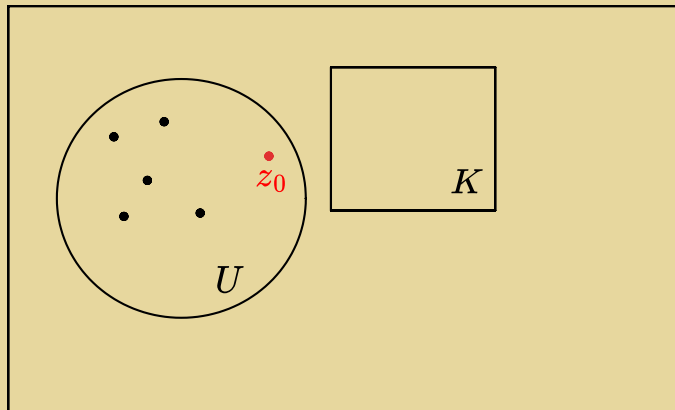
Let $f \in \mathcal{C}^{\mathcal{H}}(U)$ be a *holomorphic function* on an open subset $U \subseteq \mathbb{C}$, and let $K \subseteq \mathbb{C}$ be *compact*. Then for all $\epsilon > 0$ we have $b_1, \dots, b_n$ in $\mathbb{C}$ and $a_1, \dots, a_n$ in $\mathbb{C} \setminus K$ such that

$$\left| f(z) - \sum_{k=1}^n \frac{b_k}{z - a_k} \right| < \epsilon \text{ on } z \in K$$

poles of rational functions can be approximately pushed to any point

pushing poles around the plane

Let f be a *rational function* with poles on a connected open subset $U \subseteq \mathbb{C}$ and $K \subseteq \mathbb{C} \setminus U$ be a compact set.



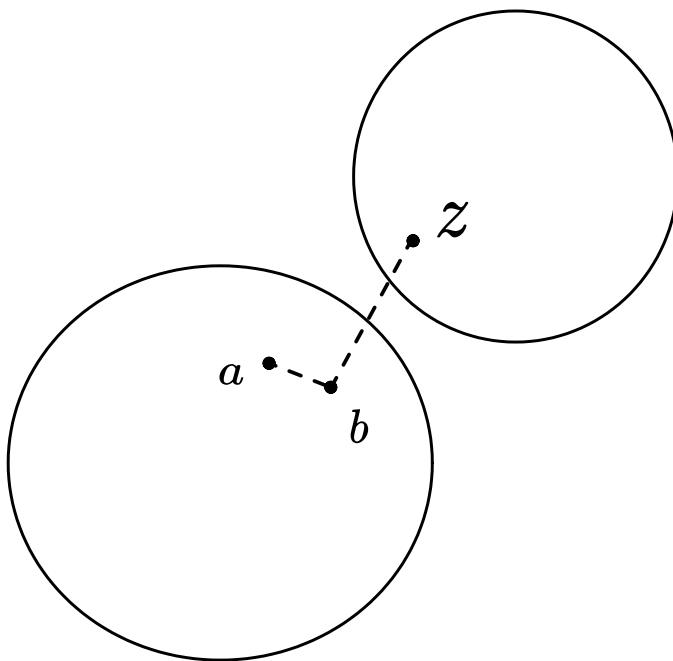
Then for all $z_0 \in U, \epsilon > 0$ there is a rational function of the form $P(z)/(z - z_0)^n$ such that

$$\left| f(z) - \frac{P(z)}{(z - z_0)^n} \right| < \epsilon \text{ on } z \in K$$

- On a compact $K \subset \mathbb{C}$, given $\epsilon > 0$, there is a rational function $\sum_{i=1}^n \frac{b_i}{z - a_i}$ where $a_i \notin K$ then

$$\left| f(z) - \sum_{i=1}^n \frac{b_i}{z - a_i} \right| < \epsilon$$

- Let U be connected open, $K \subseteq \mathbb{C} \setminus U$ compact.



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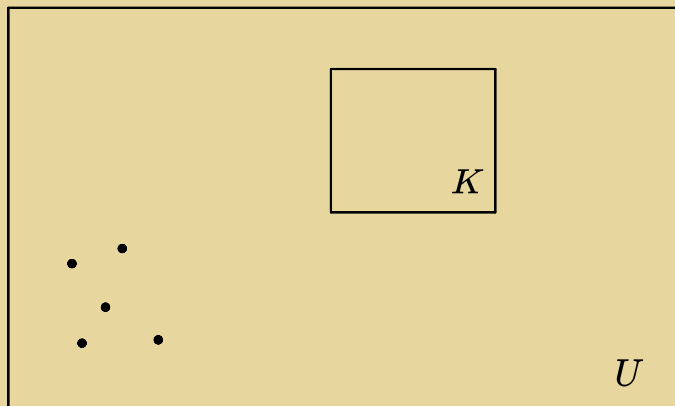
$$|a - b| < |z - b| \implies \frac{1}{z - a} = \sum_{n \geq 0} \frac{(a - b)^n}{(z - b)^{n+1}}$$

$$R_b := \{a \in U : \begin{array}{l} \text{rat with pole at } a \\ \text{can be ufm approx by rat with pole at } b \end{array}\}$$

- then R_b is open
- and $R_b \subset U$ contains all its limit points
- thus $R_b \subset U$ is open and closed, so $R_b = U$ as U is connected

pushing poles to infinity

Let f be a *rational function* with poles on a connected open subset $U \subseteq \mathbb{C}$ such that $\{z : |z| > R\} \subseteq U$ and $K \subseteq \mathbb{C} \setminus U$ be a compact set.

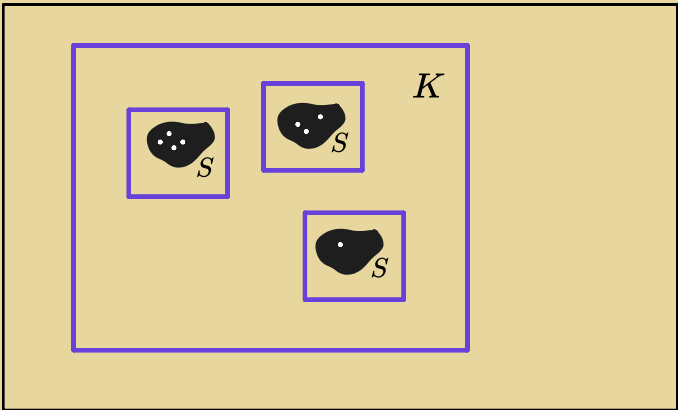


Then for all $\epsilon > 0$ there is a polynomial $P(z)$ such that

$$|f(z) - P(z)| < \epsilon \text{ on } z \in K$$

any holomorphic function can be uniformly approximated by rational functions on a compact set with poles *pushed anywhere* outside that compact set

(Runge's theorem) Let $K \subseteq \mathbb{C}$ be compact, f be holomorphic in some open interval containing K and $S \subseteq \mathbb{C} \setminus K$ which meets every connected component of $\mathbb{C} \setminus K$ except the one containing $\{z : |z| > R\}$ for some $R > 0$.



Then for every $\epsilon > 0$ there is a rational function

$$\sum_{k=1}^n P_k(z)/(z - a_k)^{n_k}$$

where P_k are polynomials, n_k are positive integers and $a_k \in S$ such that

$$\left|f(z) - \sum_{i=1}^n \frac{P_k(z)}{(z - a_k)^{n_k}}\right| < \epsilon \text{ on } z \in K$$

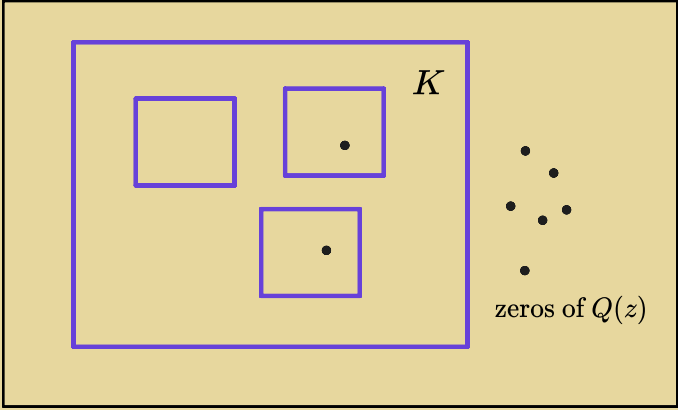
The rational function which approximates f has poles only in S . So, the statement also says if $\mathbb{C} \setminus K$ is connected, S may be empty, so the poles are at infinite/ approximation is by polynomials.

If $S \subseteq \mathbb{C} \setminus K$ meets every connected component of $\mathbb{C} \setminus K$, then

$$\forall \epsilon > 0 \left(\exists \frac{P(z)}{Q(z)}, z \in K \implies \left|f(z) - \frac{P(z)}{Q(z)}\right| < \epsilon \right)$$

and the zeroes of $Q(z)$ lies in S .

(Mergelyan's theorem) Let f be a continuous function on a compact set $K \subseteq \mathbb{C}$ which is holomorphic on the interior of K . Then for all $\epsilon > 0$, there is a rational function $P(z)/Q(z)$ where $Q(z)$ has zeros outside K



such that

$$\left|f(z) - \frac{P(z)}{Q(z)}\right| < \epsilon \text{ on } z \in K$$

We can also use the pole pushing argument as above to further control the zeros of $Q(z)$.

This result also generalises the *Weierstrass approximation theorem* in real analysis.

Current note has 0 direct children and 0 total descendants.

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And it has 19 siblings.

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 - $\mathcal{O}(D)$
 - $\mathcal{O}^p(H^2_U)$
 - $\mathcal{O} \cap L^p$

- $\mathcal{O}(S^1)$
- Zeros and singularities of holomorphic functions