Info

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Geodesics in homotopy classes

Let (M, g) be a complete, connected Riemannian manifold with $p, q \in M$. Every path-homotopy class of paths from p to q

$$[\sigma] \in \mathrm{homt}([0,1], M; \{0,1\})$$

contains a geodesic segment that minimizes length among all admissible curves in the same path homotopy class $[\sigma]$.

Let $\pi : (X, \pi^*\mathrm{g}) \to M$ be the Riemannian universal cover. Given a path $\sigma : [0,1] \to M$ from p, q has a smooth lift $\tilde{\sigma}$ starting at $\tilde{p} \in \pi^{-1}(p)$.

- From

  (Hopf-Rinow) Let (M, g) be a connected Riemannian manifold. Then M is geodesically complete

  $$\Longleftrightarrow$$

  (M, d_g) is a complete metric space

  $$\Longleftrightarrow$$

  $\exp_p : T_pM \to M$ is defined for some $p \in M$

  $$\Longleftrightarrow$$

  every closed d-bounded set in M is compact.
  If (M, g) satisfies any (hence all) of the above, then each pair of points in M can be joined by a **length-minimizing** geodesic segment. In particular, for each $p \in M$ the exponential

  $$\exp_p : T_pM \to M$$

  is surjective.

  choose the length-minimizing $\pi^*\mathrm{g}$-geodesic segment $\tilde{\gamma}$ from $\tilde{p}$ to $\tilde{\sigma}(1)$.
- As π is a local isometry $\gamma := \pi \circ \gamma$ is a geodesic in M from p, q which is in the same homotopy class as γ.
- If γ_1 is any other admissible curve from p to q in the same homotopy class as σ, ...

> **Proposition:** Let (M, g) is a compact, connected Riemannian manifold with $p_0 \in M$. Every non-trivial free homotopy class of loops
>
> $$[\sigma] : \mathrm{homt}(S^1, M) \setminus \{[p_0]\}$$
>
> contains a closed geodesic that has minimum length among all admissible loops in the same free homotopy class $[\sigma]$.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Geodesics in homotopy classes

And it has 14 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Integrating sectional curvature
        - Riemannian manifolds with negative curvature
        - Riemannian manifolds with non-positive sectional curvature
        - Cut locus in a complete Riemannian manifold
        - Exponential map for a complete Riemannian manifold
        - Geodesic completeness $\Longleftrightarrow$ metric completeness of Riemannian manifolds
        - Geodesics in homotopy classes
        - Homogeneous Riemannian manifolds
        - Homogeneous and isotropic Riemannian manifolds
        - Poles of a complete Riemannian manifold
        - Riemannian symmetric spaces
        - Uniquely geodesic complete Riemannian manifolds
        - Riemannian homogeneous manifolds where $\exp_p$ is a diffeomorphism
        - Complete Riemannian manifolds without conjugate points