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This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on November 1, 2024 7:02:10 PM,
was last modified on September 26, 2025 1:02:24 PM, and
was exported on September 7, 2026 3:31:20 PM.

Riemannian symmetric spaces

Definition. Riemannian symmetric spaces

A **Riemannian locally symmetric space** is a Riemannian manifold (M, g) such that for each $m \in M$ there is a neighborhood $U_m \subseteq M$ of p and a local isometry which is a *geodesic symmetry* or a *point reflection*

$$\begin{aligned} \sigma_m : U_m &\to U_m \\ \sigma_m(m) &= m \\ \mathfrak{D}_m \sigma_m &= -\mathsf{Id}_{T_m M} \end{aligned}$$

A **Riemannian (globally) symmetric space** is a Riemannian manifold (M, g) such that for each $m \in M$ there is a (global) isometry which is a *geodesic symmetry* or a *point reflection*

$$\begin{aligned} \sigma_m : M &\to M \\ \sigma_m(m) &= m \\ \mathfrak{D}_m \sigma_m &= -\mathsf{Id}_{T_m M} \end{aligned}$$

Every Riemannian (globally) symmetric space is Riemannian homogeneous, and thus complete.

A Riemannian manifold (M, g) is a locally symmetric space $\iff$ its Riemann curvature tensor is parallel $\nabla R_g \equiv 0$.

Let (M, g) be a complete, simply connected Riemannian manifold with parallel Riemann curvature tensor. Then (M, g) is a symmetric space.

> **Corollary:** Let (M, g) is a complete, connected Riemannian manifold. Then the following are equivalent:
>
> - (M, g) has parallel curvature tensor
> - (M, g) is a Riemannian locally symmetric space
> - $$(M, g) \cong_{\mathsf{Riem}} {}^{(X, g_X)}\!/_{\Gamma}$$
>
>   where (X, g_X) is a Riemannian (globally) symmetric space and Γ is a discrete Lie group that acts freely, properly and isometrically on (X, g_X).