Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
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$\mathrm{Sym}(2, \mathbb{R})$

The matrices of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $(a, b, c) \in \mathbb{R}^3$ form a 3 dimensional $\mathbb{R}$-vector space.

The functions

$$\mathrm{tr} = a + c$$

and

$$\det = ac - b^2$$

are linear and polynomial functions on $\mathbb{R}^3 \ni (a, b, c)$.

https://www.desmos.com/3d/roexq9eni3

eigenvalues are real and orthonormal basis exists

The roots of characteristic polynomial

$$X^2 - \mathrm{tr} X + \det = 0$$

are

$$\frac{\mathrm{tr}}{2} \pm \frac{\sqrt{\mathrm{tr}^2 - 4\det}}{2}$$

where

$$\begin{aligned} \mathrm{tr}^2 - 4\det &= (a+c)^2 - 4(ac - b^2) \\ &= a^2 + c^2 - 2ac + b^2 \\ &= (a-c)^2 + (2b)^2 \geq 0 \end{aligned}$$

So the roots are real.

$$\lambda_1, \lambda_2 = \frac{a+c}{2} \pm \frac{\sqrt{(a-c)^2 + (2b)^2}}{2}$$

Thus we get

$$\begin{aligned} \lambda_1 + \lambda_2 &= \mathrm{tr} = a + c \\ |\lambda_1 - \lambda_2| &= \sqrt{(a-c)^2 + (2b)^2} \end{aligned}$$

- If the symmetric matrix has two **distinct** eigenvalues, then it is diagonalizable. For the two eigenlines spanned by v_1, v_2 corresponding to two eigenvalues λ_1, λ_2

  $$0 = \langle Av_1, v_2 \rangle - \langle v_1, Av_2 \rangle = (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle$$

  Hence the eigenlines are orthogonal.
- Two eigenvalues are **same** if and only if

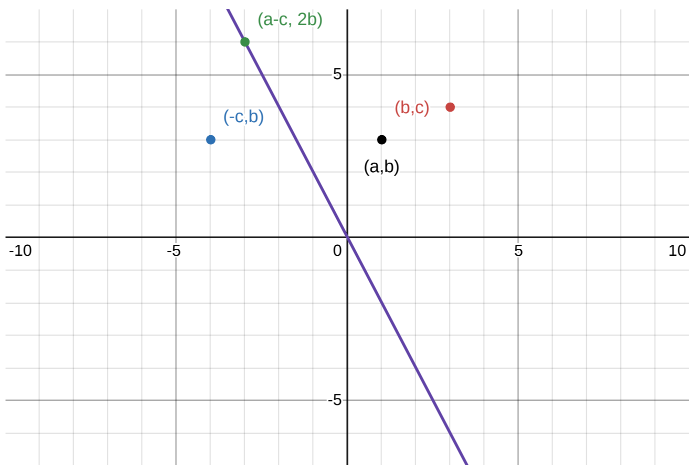
  $$(a-c)^2 + (2b)^2 = 0 \iff a = c \text{ and } b = 0$$

  which gives a scaling by $a = c$ matrix, which is diagonal. Thus, the standard orthonormal basis forms a eigenbasis.

Hence, a 2x2 symmetric matrix is always diagonalizable with a orthonormal eigenbasis

$$\{v_1, v_2 = R^{\pi/2} v_2\}$$

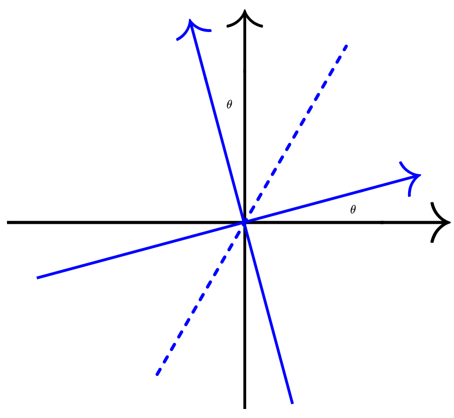
orthogonal eigenlines



We look at the point

$$\begin{aligned} \begin{bmatrix} a - c \\ 2b \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -c \\ b \end{bmatrix} \\ &= A(R^{\pi/2}(-e_2)) + R^{\pi/2}(A(e_2)) \\ &= [R^{\pi/2}, A](e_2) \end{aligned}$$

We go to an orthonormal eigenbasis by rotating by θ.



$$R^\theta \begin{bmatrix} a & b \\ b & c \end{bmatrix} R^{-\theta} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

where λ_1 is the eigenvalue of $R^\theta(e_1)$.

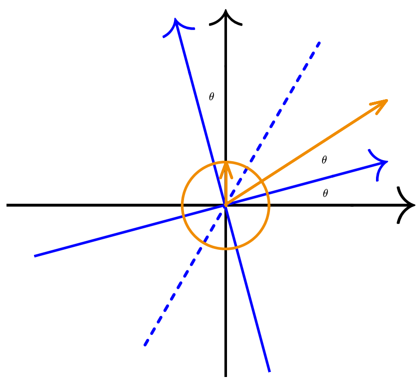
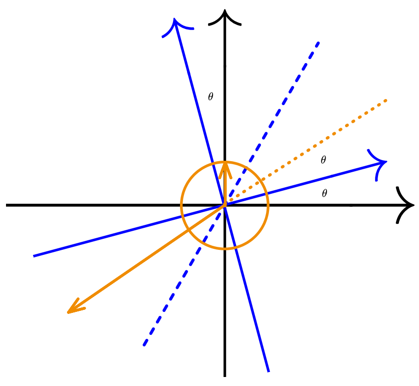
Then the commutator equation in $\{R^\theta(e_1), R^{\theta+\pi/2}(e_1)\}$-basis becomes

$$[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}] = (\lambda_1 - \lambda_2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is a scaling and reflection about the angle bisector of the eigenlines. Then

$$[R^{\pi/2}, A](e_2)$$

becomes the image of e_2 under such a transformation.

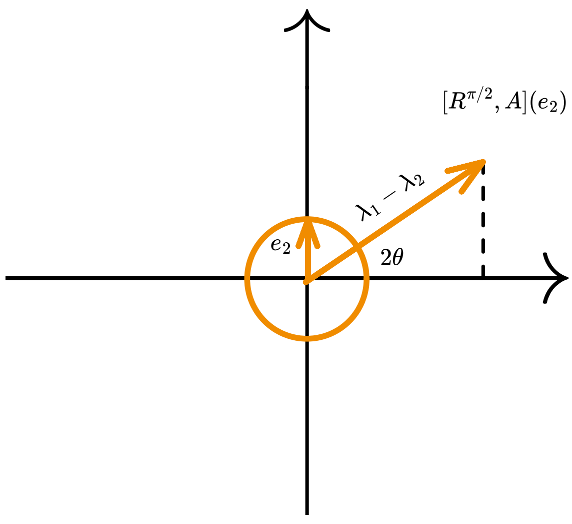



Hence, the bisector of the lines $[R^{\pi/2}, A](e_2)$ and e_1 is one of the eigenlines.

The point

$$[R^{\pi/2}, A](e_2) = \begin{bmatrix} a-c \\ 2b \end{bmatrix} = (\lambda_1 - \lambda_2) R^\theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R^{-\theta}$$

has a length of $\lambda_1 - \lambda_2$ where if we choose $\theta < \frac{\pi}{4}$ then λ_1 is the eigenvalue of the eigenline in the first quadrant.



This is a geometric restatement of

$$(\lambda_1 - \lambda_2)^2 = (a-c)^2 + (2b)^2$$

from before.

Claim: $(a-c, 2b)$ sits on the quadrant with the eigenline of **higher eigenvalue**??

$SO_{\mathbb{R}}(2)$-conjugacy action

$$\frac{\mathbb{R}}{\frac{\pi}{2}\mathbb{Z}} \times \mathbb{R}^2 \to \{\text{symmetric 2x2 } \mathbb{R}\text{-matrices}\}$$
$$(\theta, \lambda_1, \lambda_2) \mapsto R^{-\theta} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} R^\theta$$

inverse

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

idempotent

"topology of symmetric idempotent" is not created yet. Click to create.

Current note has 0 direct children and 0 total descendants.

- squishy
    - Sym
        - $\mathrm{Sym}(2, \mathbb{R})$

And it has 1 siblings.

- squishy
    - Sym
        - $\mathrm{Sym}(2, \mathbb{R})$