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Isometries of a semi-Riemannian manifold

Proposition:

- Let $(M_1, g_1), (M_2, g_2)$ be semi-Riemannian manifolds with M_1 **connected**. Suppose

$$\varphi, \psi : M_1 \rightarrow M_2$$

are local isometries such that for some $p \in M_1$ we have $\varphi(p) = \psi(p)$ and $\mathfrak{D}_p \varphi = \mathfrak{D}_p \psi$. Then $\varphi \equiv \psi$.

- Let (M, g) be a **connected** Riemannian n -manifold and

$$G \curvearrowright (M, g)$$

be a smooth, **faithful and isometric** action.

- For any $p \in M$, the orbit map

$$G/G_p \rightarrow M$$

is an immersion, so $\dim G - \dim G_p = \dim M$ and the derivative/isotopy action

$$G_p \rightarrow O(T_p M)$$

is injective.

- This implies $\dim G \leq n(n+1)/2$.

Isometry group of a Riemannian manifold

(Myers-Steenrod) Let (M, g) be a Riemannian manifold. Then $\text{Isom}(M, g)$ with the compact-open topology

Definition. Compact-open topology on Y^X

Let X be a topological space and (Y, d) be a metric space. Let $K \subseteq X$ be compact and $\epsilon > 0$ and consider the subsets

$$B_\epsilon(f, K) := \left\{ g \in Y^X \mid \sup_{x \in K} d(f(x), g(x)) < \epsilon \right\}$$

The topology on Y^X generated by these subsets is the **compact-open** topology $\mathfrak{T}_{\text{cpt}}$.

is a Lie group such that the action

$$\text{Isom}(M, g) \curvearrowright M$$

is **smooth** and **proper**.
If M is connected, we have

$$\dim \text{Isom}(M, g) \leq \frac{1}{2} \dim M (\dim M + 1)$$

with equality $\iff M$ is isometric to $\mathbb{R}^n$ or S^n or $\mathbb{R}P^n$ or $\mathbb{R}H^n$ upto scaling.
If (M, g) is complete and connected?, the action of $G \leq \text{Isom}(M, g)$ is **proper** $\iff G$ is closed in $\text{Isom}(M, g)$.

- [1]
- [2]
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- Let $f_n \in \text{Isom}(M, g)$ be such that

$$f_n \xrightarrow{\text{pointwise}, n \rightarrow \infty} f$$

- Let $x_0 \in M, \epsilon > 0$. Let $U_\epsilon := B_\epsilon(x_0)$. Then

$$\forall x \in U_\epsilon, n \in \mathbb{N}, d(f_n(x), f(x_0)) = d(x, x_0) < \epsilon$$

showing the sequence is equicontinuous

Definition. Equicontinuous families in $\text{EndTop}(X, Y)$

Let X be a topological space, (Y, d) be a metric space, and $\mathcal{F} \subseteq \text{EndTop}(X, Y)$. Then

- $\mathcal{F}$ is **equicontinuous at** $x_0 \in X$ if $\forall \epsilon > 0$ there exists an open nb $x_0 \in U \subseteq X$ such that

$$\begin{aligned} \forall x \in U, \forall f \in \mathcal{F} \quad d(f(x), f(x_0)) &< \epsilon \\ \iff \forall z \in \mathcal{F}(U), d(z, f(x_0)) &< \epsilon \end{aligned}$$

- $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at all points $x \in X$.

- For every x_0 , the sequence converges

$$\{f_n(x_0)\}$$

so it is bounded.

- Thus by

(sequential Arzela-Ascoli theorem on σ -compact and locally compact spaces onto metric spaces) Let X be a σ -compact and locally compact topological space and (Y, d) be a metric space. Then a **equicontinuous and**

pointwise bounded subset

$$\mathcal{F} \subseteq (\text{EndTop}(X, Y), \mathfrak{T}_{\text{cpt}})$$

is sequentially pre-compact (any sequence in $\mathcal{F}$ always has a subsequence in $\mathcal{F}$ that converges uniformly on compact subsets of X continuous map $X \rightarrow Y$).

f_n converges to f uniformly on compact subsets.

- Now

$$d(x, y) = d(f_n(x), f_n(y)) \xrightarrow{n \rightarrow \infty} d(f(x), f(y))$$

thus $f \in \text{Isom}(M, g)$.

Let $x_n \xrightarrow{n \rightarrow \infty} x$ and $g_n(x_n) \xrightarrow{n \rightarrow \infty} y$.

- ...

Corollaries:

Let (M, g) be a Riemannian manifold and $\text{Isom}(M, g)$ be its isometry group and $H \leq \text{Isom}(M, g)$ be a closed subgroup.

- For each $x \in M$, the stabilizer/isotropy group $\text{Isom}_x(M, g)$ is **compact**
 - thus

$$\text{Isom}_x(M, g) \rightarrow O(T_x M)$$

is an *embedding*

- For $x \in M$, the H -orbit $H\{x\}$ of x is a *properly embedded submanifold*, as in, the orbit map

$$H/H_x \rightarrow H\{x\}$$

is a **proper embedding**, where

$$H_x = \{h \in H | hx = x\} = \underbrace{\text{Isom}_x(M, g)}_{\text{compact}} \cap H$$

$\implies H_x$ is **compact**.

- Thus if an orbit $H\{x\}$ is compact, then H is compact.
- If $Hx = x \iff H_x = H \implies H$ is compact.
- In particular if the isometry group $\text{Isom}(M, g)$ has a fixed point, then $\text{Isom}(M, g)$ must be compact.
- If M is compact then $\text{Isom}(M, g)$ is also compact.
 - The converse is in general not true?
 - However, if $\text{Isom}(M, g)$ acts transitively and is compact, then M is also compact.

Closed subgroups of $\text{Isom}(M, g)$

	not proper action/not closed subgroup of $\text{Isom}(M, g)$	proper action/closed subgroup
each stabilizer is compact/finite	dense subgroup of $T^2 \curvearrowright T^2$	any closed subgroup
transitive, with compact stabilizer	possible!	...

Closed transitive subgroups of $\text{Isom}(M, g)$

Which groups appear as $\text{Isom}(M, g)$?

[6]

Current note has 0 direct children and 0 total descendants.

- [structures based on sets](#)
 - [manifolds](#)
 - [ℝ-manifold semi-Riemannian](#)
 - [Isometries of a semi-Riemannian manifold](#)

And it has 18 siblings.

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 - [Vector fields on Riemannian manifolds](#)

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1. [Tammo tom Dieck - Transformation Groups-De Gruyter \(1987\)](#) ↩
 2. [Marcos M Alexandrino; Renato Ghini Bettiol - Lie Groups and Geometric Aspects of Isometric Actions \(2015\),_page 77](#) ↩
 3. [PROPER ISOMETRIC ACTIONS 0811.0547v1.pdf](#) ↩
 - 6-28. Suppose (M, g) is a complete, connected Riemannian n -manifold whose isometry group is a Lie group acting smoothly on M . (The Myers–Steenrod theorem shows that this is always the case when M is connected, but we have not proved this.) Prove that if G is a closed subgroup of $\text{Iso}(M)$, then G acts properly on M , using the following outline. Suppose (φ_m) is a sequence in G and (p_m) is a sequence in M such that $p_m \rightarrow p_0$ and $\varphi_m(p_m) \rightarrow q_0$.
 - (a) Prove that $\varphi_m(p_0) \rightarrow q_0$.
 - (b) Let $B_r(p_0)$ be a geodesic ball centered at p_0 ; let $0 < \varepsilon < r$; let $(b_1, \dots, b_n)$ be an orthonormal basis for $T_{p_0}M$; and let $p_i = \exp_{p_0}(\varepsilon b_i)$ for $i = 1, \dots, n$. Prove that there exist a linear isometry $A: T_{p_0}M \rightarrow$
 4. ↩
 5. [Peter Petersen - Riemannian Geometry_\(2016\),_p.227](#) ↩
 6. [dg.differential.geometry - Can every Lie group be realized as the full isometry_group of a Riemannian manifold? - MathOverflow](#) ↩