Info

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Conical limit set

Definition. Conical limit point

A point $\xi \in \partial_\infty \mathbb{R}\boldsymbol{H}^n$ is a **conical limit point** of $G \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ if $\exists x \in \mathbb{R}\boldsymbol{H}^n$ and $\{g_n\} \subseteq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$

- and there exists a geodesic ray γ with $\gamma(\infty) = \xi$ such that
$$g_n x \xrightarrow{n\to\infty} \xi$$
$$\exists r > 0 : \{g_n x\} \subseteq B_r(\gamma)$$
- $\iff$
$$\left\{ \frac{|\xi - g_n(x)|}{1 - |g_n(x)|} \right\} \text{ is bounded}$$
(in the ball model B^n)

The set of all such conical limit points is the **conical limit set**
$$\Lambda_c(\Gamma) \subseteq \Lambda(\Gamma)$$

- For a countable dense subset $\{x_n\} \subseteq B^n$ we have
$$\Lambda_c(\Gamma) = \bigcap_{n\geq 1} \bigcup_{k>0} L(x_n, k, 1)$$

> **Proposition:** Let $\Gamma <_d \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$. Then Γ converges at $n-1 \iff m(\Lambda_c(\Gamma)) = 0$.

> - For a countable dense subset $\{x_n\} \subseteq B^n$ we have
> $$\Lambda_c(\Gamma) = \bigcap_{n\geq 1} \bigcup_{k>0} L(x_n, k, 1)$$
>
> and
>
> > **Proposition:** Let Γ converges at $(n-1)\alpha$. Then
> > $$m_{S^{n-1}}\left(\bigcup_{k>0} L(a, k, \alpha) \right) = 0$$

Let $\Gamma = \{\gamma_n\}$ and $x_n := \gamma_p^{-1}(o)$. Then
$$\Lambda_c(\Gamma) = \bigcup_{\delta>0} \bigcap_{N\geq 1} \bigcup_{p\geq N} b(o, x_p, \delta)$$

...

> **Proposition:** Let Γ be cocompact. Then $\Lambda(\Gamma) = \Lambda_c(\Gamma) = \partial\mathbb{R}\boldsymbol{H}^n$.

There exists $\epsilon > 0$ such that $D(\Gamma) \subseteq B_{1-\epsilon}(o)$.

- For any $\xi \in \partial\mathbb{R}\boldsymbol{H}^n$ and σ be the ray extending $\overline{o\xi}$.
- $$\{\gamma(D) | \gamma(D) \cap \sigma \neq \emptyset\}$$
  is infinite
- ...

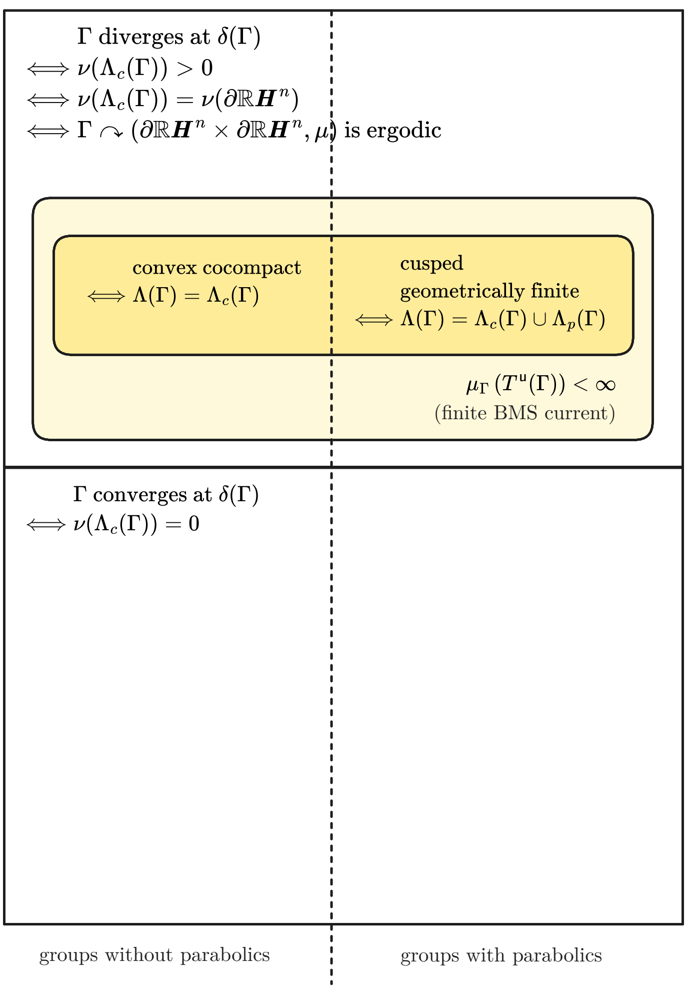
Let Γ be convex cocompact. Then $\Lambda(\Gamma) = \Lambda_c(\Gamma)$.

Let $\xi \in \Lambda(\Gamma), x \in \mathrm{convex}(\Lambda)$ and σ be the ray extending $\overline{x\xi}$. Then $\sigma \subseteq \mathrm{convex}(\Lambda)$.

- ...

> **Proposition:** Let ν be a Γ-conformal density of dimension α.
>
> 1. $$\Gamma \text{ converges at } \alpha \implies \nu(\Lambda_c(\Gamma)) = 0$$
> $$\iff \Gamma \text{ diverges at } \alpha \implies \nu(\Lambda_c(\Gamma)) > 0$$
> 2. $$\nu(\Lambda_c(\Gamma)) > 0 \implies \nu(\Lambda_c(\Gamma)) = 1$$
> 3. $\nu(\Lambda_c(\Gamma)) = 1 \iff$ geodesic flow is completely conservative
> 4. $$\Gamma \text{ converges at } \alpha \implies \nu(\Lambda_c(\Gamma)) = 0$$
> 5. geodesic flow is completely conservative $\implies$ it is ergodic
> 6. geodesic flow is ergodic $\iff \Gamma \curvearrowright (\partial^2\mathbb{R}\boldsymbol{H}^n, \nu \otimes \nu|_{\partial^2\mathbb{R}\boldsymbol{H}^n})$ is ergodic
> 7. $\Gamma \curvearrowright (\partial^2\mathbb{R}\boldsymbol{H}^n, \nu \otimes \nu|_{\partial^2\mathbb{R}\boldsymbol{H}^n})$ is ergodic $\implies \Gamma \curvearrowright (\mathbb{R}\boldsymbol{H}^n \times \mathbb{R}\boldsymbol{H}^n, \nu \otimes \nu)$ is ergodic

$$\begin{aligned}&\Gamma \text{ diverges at } \alpha\\ \iff& \nu(\Lambda_c(\Gamma)) > 0\\ \iff& \nu(\Lambda_c(\Gamma)) = 1\\ \iff& \Gamma \curvearrowright (\partial^2\mathbb{R}\boldsymbol{H}^n, \nu\otimes\nu|_{\partial^2\mathbb{R}\boldsymbol{H}^n}) \text{ is ergodic}\end{aligned}$$



Proposition: $\Gamma \curvearrowright (\partial\mathbb{R}\boldsymbol{H}^n, [\nu])$ is ergodic $\iff$ ν is the unique Γ-conformal density of dimension $\dim\nu$ (based at the base point of ν) upto scaling by a constant.

Proposition: Γ converges at $\alpha \implies \nu^\alpha(\Lambda_c(\Gamma)) = 0$

[1]

Proposition: $\nu(\Lambda_c(\Gamma))$ is either 0 or of full measure

[2]

Proposition:

- $\Lambda_c(\Gamma)$ has full ν-measure $\iff$ g^t is conservative
- $\nu(\Lambda_c(\Gamma)) = 0 \iff g^t$ is completely dissipative

By

> **Proposition:** $\nu(\Lambda_c(\Gamma))$ is either 0 or of full measure

Proposition: Γ diverges at $\alpha \iff g^t$ is conservative.

($\implies$) By

> **Proposition:** Γ converges at $\alpha \implies \nu^\alpha(\Lambda_c(\Gamma)) = 0$

and

> **Proposition:**
> - $\Lambda_c(\Gamma)$ has full ν-measure $\iff$ g^t is conservative
> - $\nu(\Lambda_c(\Gamma)) = 0 \iff g^t$ is completely dissipative

($\impliedby$)

- $$\nu(\Sigma_1(\gamma)) \geq \epsilon$$
- $$\begin{aligned}\nu(\Sigma_2(\gamma)) &= \nu(\gamma(\gamma^{-1}\Sigma_2(\gamma)))\\ &= \int_{\xi\in\gamma^{-1}\Sigma_2(\gamma)} b(\xi)^\alpha \mathrm{d}\nu\\ ? &> \lambda(\Gamma)\exp(-\alpha d(o,\gamma(o)))\end{aligned}$$
- Let $J := \Delta \times S^{n-1} \subseteq T^u\mathbb{R}\boldsymbol{H}^n$. Then
$$\mu(J \cap g^{-d(o,\gamma(o))}\gamma(J)) > \lambda_1 \exp(-\alpha d(o,\gamma(o)))$$
- $$\begin{aligned}E_n &:= J \cap g^{-n}\Gamma(J)\\ \mu(E_n) &= \mu(J\cap g^{-n}\Gamma(J))\\ \mu(E_{n+m}\cap E_n) &= \mu(J\cap g^{-n-m}\Gamma(J)\cap g^{-n}\Gamma(J))\end{aligned}$$
- ...

Proposition: 8.3.2 g^t conservative $\iff$ g^t ergodic

Proposition: g^t ergodic $\iff$ $\Gamma \curvearrowright (\partial\mathbb{R}\boldsymbol{H}^n \times \partial\mathbb{R}\boldsymbol{H}^n, \nu\otimes\nu)$ is ergodic

$\Gamma \curvearrowright (\mathbb{R}\boldsymbol{H}^n \times \mathbb{R}\boldsymbol{H}^n, \nu\otimes\nu)$ is not ergodic

- $\iff$ there is a subset $Q \subseteq \partial\mathbb{R}\boldsymbol{H}^n \times \partial\mathbb{R}\boldsymbol{H}^n$
$$Q := \left\{\pi(A) \quad \text{for } g^t, \Gamma\text{-invariant subset } A \subseteq\right.$$
which is Γ-invariant and $\nu\otimes\nu(Q) > 0$
- $\iff$ there is a subset
$$A := Q \times \mathbb{R}$$
which is g^t-invariant, Γ-invariant and $\nu\otimes\nu\otimes m(A) > 0$

Current note has 0 direct children and 0 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Limit sets of discrete subgroups
        - Conical limit set

And it has 1 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Limit sets of discrete subgroups
        - Conical limit set

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1. (4.4.1) ↩
2. (Theorem 4.4.) ↩