

Info

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on February 1, 2024 11:36:23 AM, was last modified on May 17, 2026 7:36:37 PM, and was exported on September 7, 2026 3:09:53 PM.

Volume change by flows on $\mathbb{R}^n$

Consider a **smooth** flow φ^t on some open subset of $\mathbb{R}^n$, this means

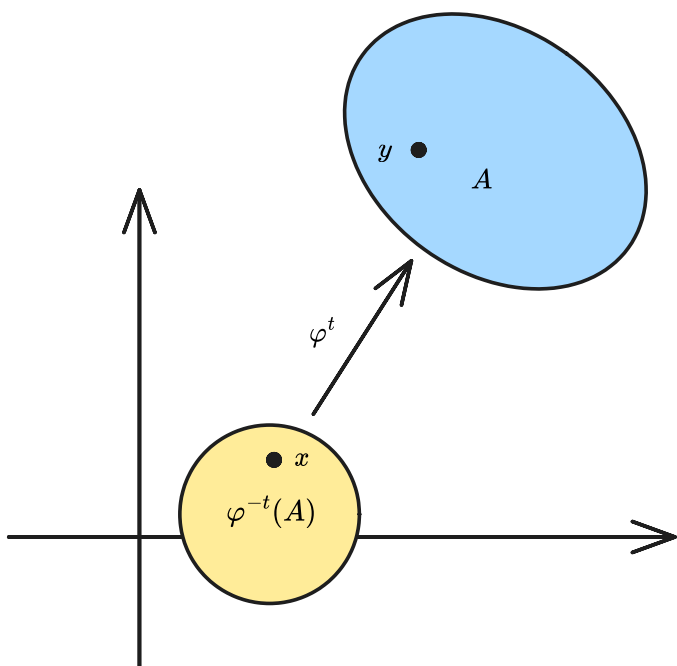
$$\varphi^t : U_t \subseteq U \rightarrow \mathbb{R}^n$$

is an injective local diffeomorphism. The *only* problem is surjectivity!

Then choose $A \subseteq \mathbb{R}^n$ so that it's preimage is not empty, then

$$\varphi^t : \varphi^{-t}(A) \rightarrow A$$

is a diffeomorphism.



We wish to compare volume of A with volume of $\varphi^{-t}(A)$. Thus we calculate volume

- By *change of variables* we get

$$\begin{aligned} \mu(A) &= \int_{y \in A} dy_1 \wedge \dots \wedge dy_n \\ &= \int_{x \in \varphi^{-t}(A)} d\varphi_1^t(x) \wedge \dots \wedge d\varphi_n^t(x) \\ &= \int_{x \in \varphi^{-t}(A)} (\det \mathfrak{D}\varphi^t) dx_1 \wedge \dots \wedge dx_n \end{aligned}$$

- So

$$\mu(\varphi^{-t}(A)) - \mu(A) = \int_{x \in A} (1 - \det \mathfrak{D}\varphi^t) dx_1 \wedge \dots \wedge dx_n$$

- Now

$$\varphi^0 = \text{Id} \implies \det \mathfrak{D}\varphi^0 = 1$$

which means

$$\lim_{t \rightarrow 0} \frac{1}{t} (1 - \det \mathfrak{D}\varphi^t) = \frac{d}{dt} \Big|_{t=0} \det \mathfrak{D}\varphi^t$$

- Thus

$$\frac{d}{dt} \Big|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x \in A} \left(\frac{d}{dt} \Big|_{t=0} \det \mathfrak{D}\varphi^t \right) dx_1 \wedge \dots \wedge dx_n$$

So rate of change of volume boils down to $\frac{d}{dt} \Big|_{t=0} \det \mathfrak{D}\varphi^t$.

computing derivative of $\det \mathfrak{D}\varphi^t$ using wedge products

🐛 Bug

$$\begin{aligned} \frac{d}{dt} \det(\mathfrak{D}\varphi^t) &= \frac{d}{dt} \left(\frac{\partial \varphi^t}{\partial x_1} \wedge \dots \wedge \frac{\partial \varphi^t}{\partial x_n} \right) \\ &= \frac{\partial}{\partial x_1} \underbrace{\frac{d}{dt} \varphi^t}_{X(\varphi^t(x))} \wedge \frac{\partial \varphi^t}{\partial x_2} \wedge \dots \wedge \frac{\partial \varphi^t}{\partial x_n} + \dots \\ &= \left(\mathfrak{D}X \left(\frac{\partial}{\partial x_1} \varphi^t \right) \right) \wedge \dots \\ &= \underbrace{\frac{\partial X_1}{\partial x_1} \det \mathfrak{D}\varphi^t + \dots}_{\det(\mathfrak{D}\varphi^t) \text{div} X} \end{aligned}$$

computing derivative of $\det \mathfrak{D}\varphi^t$ using Jacobi's formula

From

☰ (Jacobi's formula)

$$\frac{d}{dt} \det A(t) = (\det A(t)) \text{tr} \left(A(t)^{-1} \frac{d}{dt} A(t) \right)$$

and

$$\begin{aligned} A'(t) &= B(t)A(t) \\ \implies (\det A)'(t) &= \det(A(t)) \text{tr}(A(t)^{-1} B(t) A(t)) \\ &= \det(A(t)) \text{tr}(B(t)) \end{aligned}$$

along with

$$\begin{aligned}\frac{\partial}{\partial t}\varphi(t,x) &= X_{\varphi(t,x)} \\ \implies \frac{\partial}{\partial t}\mathfrak{D}\varphi(t,x) &= \mathfrak{D}X_{\varphi(t,x)} \\ &= \mathfrak{D}_{\varphi(t,x)}X \circ \mathfrak{D}\varphi(t,x)\end{aligned}$$

we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}\varphi^t &= \mathfrak{D}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t = \mathfrak{D}X_{\varphi^t} = \mathfrak{D}X \circ \mathfrak{D}\varphi^t \\ \implies \mathrm{tr}(\mathfrak{D}\varphi^{-t} \circ \mathfrak{D}X \circ \mathfrak{D}\varphi^t) &= \mathrm{tr}(\mathfrak{D}X)\end{aligned}$$

So we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\det \mathfrak{D}\varphi^t_{(t,x)} &= \mathrm{tr}(\mathfrak{D}X)_{(\varphi^t(x))} \det \mathfrak{D}\varphi^t_{(t,x)} \\ &= \mathrm{div}(X)_{(\varphi^t(x))} \det \mathfrak{D}\varphi^t_{(t,x)}\end{aligned}$$

- At $t = 0$, $\varphi^0 = \mathrm{Id}$ thus we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det \mathfrak{D}\varphi^t\right)(x) = \mathrm{div}(X)_x$$

- In general

$$\det \mathfrak{D}\varphi^t_{(t,x)} = \exp\left(\int_{[0,t]} \mathrm{div}(X)_{\varphi^t(x)}\mathrm{d}t\right)$$

[1]

[2]

computing derivative of $\det \mathfrak{D}\varphi^t$ using Liouville's formula

[3]

rate of change of volume is determined by integral of divergence

Hence, we have

☰ For flow

$$\exp(tX) =: \varphi^t$$

of a smooth vector field X on a open subset of $\mathbb{R}^n$, the rate of change of volume of $\phi^{-t}(A), A \subseteq \mathbb{R}^n$ at $t = 0$ is the volume integral of divergence of X , on A

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x \in A} \mathrm{div}(X)_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Thus

- $\mathrm{div}(X) \equiv 0 \iff$ the flow of X is *volume preserving*

using Reynolds transport theorem - Wikipedia

Current note has 0 direct children and 0 total descendants.

- stem
 - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
 - Volume change by flows on $\mathbb{R}^n$

And it has 10 siblings.

- stem
 - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
 - Constant of a flow in $\mathbb{R}^n$
 - Euler's iterative solution method for first order ODEs
 - Existence of integral curves of vector fields in $\mathbb{R}^n$
 - Fixed points
 - Flow of a vector field in $\mathbb{R}^n$
 - Graph $\rightarrow$ polynomial ODE
 - Gradient flows on $\mathbb{R}^n$
 - Hamiltonian vector fields in $\mathbb{R}^{2n}$ with standard symplectic form
 - Probability distribution of flow
 - Volume change by flows on $\mathbb{R}^n$

1. <https://math.stackexchange.com/a/3456775/1290493> ↩
2. <https://mathoverflow.net/a/327773> ↩
3. <https://math.stackexchange.com/a/312623/1290493> ↩