

Info

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$$\widehat{\cdot} : L^2[0, 1] \cong_{\text{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$$

(Parseval's and Riesz-Fischer theorems) Let $f \in L^2[0, 1]$. Then the partial Fourier summation converges to f almost everywhere and in L^2

$$S_N f \xrightarrow{N \rightarrow \infty} f \quad \text{ae and } L^2$$

and the Fourier transform is a $\mathbb{C}$ -linear bijection

$$\widehat{\cdot} : L^2[0, 1] \cong_{\text{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$$

which satisfies $\|f\|_2 = \|\hat{f}\|_2$ (is an isometry).

trigonometric series for an l^2 sequence

- Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$ and consider

$$T_N(x) := \sum_{-N \leq n \leq N} c_n e^{2\pi i n x}$$

- The series satisfies

$$\|T_N - T_M\|_2^2 = \sum_{M < m \leq N} |c_m|^2$$

and thus $\{T_N\}$ is a Cauchy sequence in $L^2[0, 1]$.

- By completeness of $L^2[0, 1]$, there is a limit

$$T := \lim_{N \rightarrow \infty} T_N$$

that means

$$\|T - T_N\|_2 \xrightarrow{N \rightarrow \infty} 0$$

- Fix $n \in \mathbb{Z}$. For $N > |n|$ we have

$$\begin{aligned} \hat{T}(n) - \hat{T}_N(n) &= \int_{[0,1]} (T - T_N) e_n \\ \implies |\hat{T}(n) - c_n| &\leq \|T - T_N\|_1 \\ &\leq \|T - T_N\|_2 \\ &\xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

- Therefore,

$$\hat{T}(n) \equiv c_n$$

Hence,

Proposition: Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then

$$\left(\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \right) \in L^2[0, 1]$$

and it's Fourier transform is c itself

$$\left(\widehat{\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}} \right)(n) \equiv c_n$$

that is

$$\sum_{n \in \mathbb{Z}} (\cdot)_n e^{2\pi i n x} : l^2(\mathbb{Z}, \mathbb{C}) \rightarrow L^2[0, 1]$$

when composed with the Fourier transform is $\text{Id}_{l^2(\mathbb{Z}, \mathbb{C})}$.

Fourier transform of $L^2[0, 1]$

- Let $f \in L^2[0, 1]$ and consider a trigonometric polynomial

$$\sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta}$$

- Then

$$\begin{aligned} &\left\| f - \sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta} \right\|_2^2 \\ &= \|f\|_2^2 - \left\langle f, \sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta} \right\rangle \\ &\quad - \left\langle \sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta}, f \right\rangle + \left\| \sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta} \right\|_2^2 \\ &= \|f\|_2^2 - \sum_{-N \leq n \leq N} \hat{f}(n) \overline{b_n} - \sum_{-N \leq n \leq N} \overline{\hat{f}(n)} b_n + \sum_{-N \leq n \leq N} |b_n|^2 \\ &= \sum_{-N \leq n \leq N} |b_n - \hat{f}(n)|^2 + \|f\|_2^2 - \sum_{-N \leq n \leq N} |\hat{f}(n)|^2 \end{aligned}$$

- Then the trigonometric polynomial for which the *mean square error*

$\left\| f - \sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta} \right\|_2^2$ is mimum must be when

$$b_n \equiv \hat{f}(n)$$

And then

$$\left\|f - \sum_{-N \leq n \leq N} \hat{f}(n)e^{-2\pi i n \theta}\right\|_2^2 = \|f\|_2^2 - \sum_{-N \leq n \leq N} \left|\hat{f}(n)\right|^2$$

- In particular, because $\| \cdot \|_2 \geq 0$ we have

$$\sum_{-N \leq n \leq N} \left|\hat{f}(n)\right|^2 \leq \|f\|_2^2$$

for every $N \geq 0$.

- This implies the sum

$$\left\|\hat{f}\right\|_2^2 = \sum_{n \in \mathbb{Z}} \left|\hat{f}(n)\right|^2$$

converges and **(Bessel's inequality)**

$$\left\|\hat{f}\right\|_2^2 \leq \|f\|_2^2$$

- As

$$\hat{f} \in l^2(\mathbb{Z}, \mathbb{C})$$

for $c_n \equiv \hat{f}(n)$, from

Proposition: Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then

$$\left(\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}\right) \in L^2[0, 1]$$

and it's Fourier transform is c itself

$$\widehat{\left(\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}\right)}(n) \equiv c_n$$

that is

$$\sum_{n \in \mathbb{Z}} (\cdot)_n e^{2\pi i n x} : l^2(\mathbb{Z}, \mathbb{C}) \rightarrow L^2[0, 1]$$

when composed with the Fourier transform is $\text{Id}_{l^2(\mathbb{Z}, \mathbb{C})}$.

we conclude

$$\widehat{\left(\sum_{m \in \mathbb{Z}} \hat{f}(m) e_m\right)}(n) \equiv \hat{f}(n)$$

- Therefore,

$$\left(\sum_{m \in \mathbb{Z}} \hat{f}(m) e_m\right) \equiv f \qquad \text{ae on } [0, 1]$$

and thus

$$S_N f = \sum_{-N \leq m \leq N} \hat{f}(m) e_m \xrightarrow{N \rightarrow \infty} f \qquad \text{ae and } L^2$$

- In particular, taking $\| \cdot \|_2$ of the converging sequence implies

$$\sum_{-N \leq n \leq N} \left|\hat{f}(n)\right|^2 \xrightarrow{N \rightarrow \infty} \left\|\hat{f}\right\|_2^2 = \|f\|_2^2$$

Current note has 0 direct children and 0 total descendants.

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 - [Fourier](#)
 - $\frown : L^2[0, 1] \cong_{\text{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$

And it has 10 siblings.

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