Info

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Monotone functions on $\mathbb{R}$

Definition. Motonone functions on $\mathbb{R}$

Let $f : I\ (\text{open interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then

- f is **monotonically increasing** on I if for each $x, y \in I$ $x \geq y$ implies $f(x) \geq f(y)$
- f is **strictly increasing** on I if for each $x, y \in I$ $x > y$ implies $f(x) > f(y)$
  - $\iff$ for $c \in I$, $f(x) - f(c)$ has the same sign as $x - c$ for all $x \in I$

differentiable maps and monotonicity

Let $f : I\ (\text{open interval}) \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable. Then f is strictly monotonic $\iff$ $Z(f')$ does not contain an open interval/has empty interior/every stationary point is isolated and
f' does not change sign on $I \setminus Z(f')$.

($\implies$)

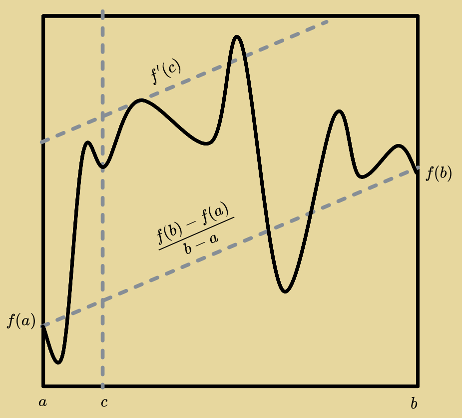
- If $Z(f')$ contains an open interval, then by

  (Lagrange's mean value theorem) Let a continuous function

  $$f : [a, b] \to \mathbb{R}$$

  be differentiable on (a, b). Then there is a point $c \in (a, b)$ at which

  $$f(b) - f(a) = (b - a)f'(c)$$

  f is constant on that interval. Hence, it is not strictly monotonic.
- Let f is strictly decreasing. Then by

  (Strictly positive derivative $\implies$ locally (strictly) increasing) Let $f : (a, b) \to \mathbb{R}$ be differentiable at $c \in (a, b)$ and

  $$f'(c) > 0 \text{ or } +\infty$$

  then there is a $I(c) \subseteq (a, b)$ in which $f(x) - f(c)$ has same sign as $x - c$, that is, f is strictly increasing on $I(c)$.

  $f' < 0$ on $I \setminus Z(f')$.

($\impliedby$) Let $f' \geq 0$. Then by

Hypothesis on f'	**Consequence of** **Corollary of mean value theorem: If** $f : (a, b) \to \mathbb{R}$ **is differentiable, then for any proper subinterval** $(x_1, x_2) \subset (a, b)$ **by** mean value theorem **we have** $\exists x \in (x_1, x_2) : f(x_2) - f(x_1) = (x_2 - x$
$f' \geq 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \geq 0$ thus f is **increasing** on (a, b)
$f' = 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) = 0$ thus f is **constant** on (a, b)
$f' \leq 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) \leq 0$ thus f is **decreasing** on (a, b)
$f' < 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) < 0$ thus f is **strictly decreasing** on (a, b)
$f' > 0 \text{ on } (a, b)$	$\forall x_1 < x_2 \in (a, b), f(x_1) - f(x_2) > 0$ thus f is **strictly increasing** on (a, b)
$f' = \alpha$ is a constant on (a, b) or $f'' = 0$ on (a, b)	$f(x) - \alpha x$ has zero derivative, thus f is **linear** $f(x) \equiv f'(0)x + f(0)$

we conclude f is increasing on I.

- If there are $x, y \in I$ such that $f(x) = f(y)$, then

$$\begin{aligned} f &\equiv f(x) \text{ on } [x. y] \\ \implies f' &\equiv 0 \text{ on } [x, y] \end{aligned}$$

- So if $Z(f')$ are isolated, we conclude that f must be **strictly increasing**.

Let $A \subseteq [0,1]$ be a compact subset with empty interior. Then there exists a $f \in \mathcal{C}^1[0,1]$ such that $Z(f') = A$.

Let the connected components of $[0,1] \setminus A$ be

$$I_1, \ldots$$

- Then

$$I_n \cap (0,1) = (a_n, b_n)$$

- ...

[1]

Current note has 1 direct children and 1 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Monotone functions on $\mathbb{R}$
      - Asymptotics of monotonically increasing functions $[0,\infty] \to [0,\infty]$

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices
    - Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
    - Discrete subgroups of $\mathbb{R}^n$
    - Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
    - Ramified germs of smooth and holomorphic functions
    - Open subsets of $\mathbb{R}^n$
    - Open subsets of $\mathbb{R}^n$ equipped with the flat metric
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
    - Quasi-analytic smooth functions on $\mathbb{R}$
    - Star-shaped subsets of $\mathbb{R}^n$
    - ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$

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1. https://math.stackexchange.com/a/1845973/1290493 ↩