ⓘ Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
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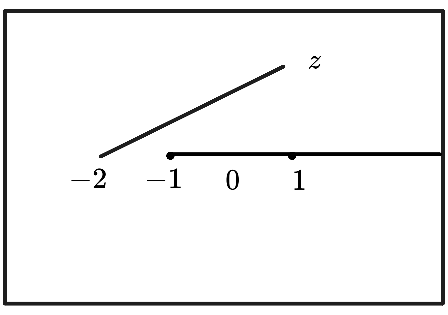
$\sqrt{z^2-1}$

Let

$$f(z) := z^2 - 1$$

then we have its logarithm on a simply connected open subset of $\mathbb{C}$



where f is non-zero

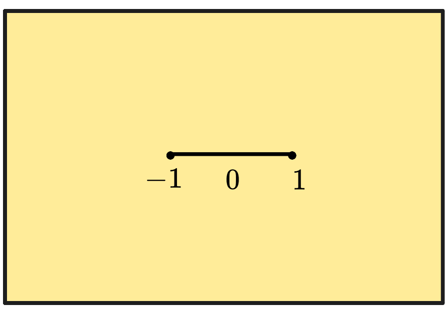
$$\begin{aligned} \mathbb{C} \setminus [-1, \infty) &\to \mathbb{C} \\ \log(f(z)) &:= \int_{-2 \mapsto z} \frac{f'(z)}{f(z)} \mathrm{d}w + \log(3) \end{aligned}$$

and we have its square root

$$\rho_+(z) := \exp\left(\frac{1}{2}\log(f(z))\right)$$

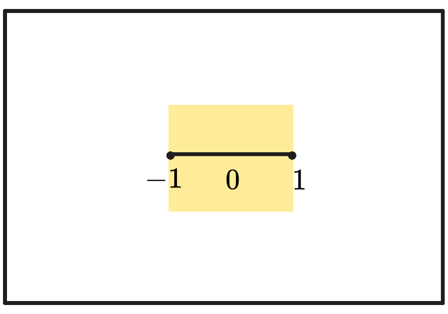
which extends continuously to $\mathbb{C} \setminus [-1, 1]$ [1].

So we have the functions

- 

  $$\rho_\pm : U_\pm := \mathbb{C} \setminus [-1, 1] \to \mathbb{C}$$

  where $\rho_- := -\rho_+$ and thus $\rho_\pm^2(z) = f(z)$

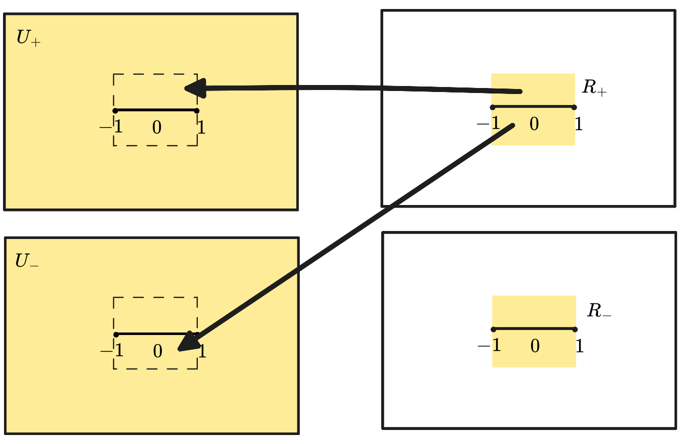
- 

  $$\varrho_\pm : R_\pm := (-1, 1) \times (1, 1) \to \mathbb{C}$$

  [2]

  and the gluing maps

$$\begin{aligned} \varphi_+ : R_+ \setminus [-1, 1] &\to U_+ \bigsqcup U_- \\ z &\mapsto \begin{cases} (U_+, z) & \Im(z) > 0 \\ (U_-, z) & \Im(z) < 0 \end{cases} \end{aligned}$$



Therefore we now have constructed the *unramified Riemann surface* of the holomorphic function $\sqrt{z^2-1}$

> **Proposition:** We obtain a Riemann surface and commutative diagram of holomorphic functions:
>
> $$\begin{array}{ccc} X_3(\sqrt{z^2-1}) := \dfrac{U_+ \bigsqcup U_- \bigsqcup R_+ \bigsqcup R_-}{\varphi_+, \varphi_-} & \xrightarrow{\rho} & \mathbb{C} \setminus \{0\} \\ \downarrow \pi & & \\ \mathbb{C} \setminus \{-1, 1\} & & \end{array}$$

We may add the points $-1, 1$ to get one step closer to the *ramified Riemann surface*

$$\begin{array}{ccc} X_2(\sqrt{z^2-1}) & \xrightarrow{\rho} & \mathbb{C} \\ \downarrow \pi & & \\ \mathbb{C} & & \end{array}$$

and even further add points at infinity to get the *ramified Riemann surface* X.

the 1-form $\frac{\mathrm{d}z}{\rho(z)}$

> **Proposition:** On the Riemann surface X_3 the function $\rho(z)$ is non-zero, and
>
> $$\frac{\mathrm{d}z}{\rho(z)}$$
>
> is a well-defined holomorphic 1-form which extends holomorphically to X.

Current note has 0 direct children and 0 total descendants.

- squishy
  - pow
    - $\sqrt{z^2-1}$

And it has 28 siblings.

- squishy
  - pow
    - Angle function
    - Bessel functions
    - $$\cos\sqrt{z}$$
    - The complex logarithm
    - $\sum_{n\geq 0} z^{n!}$
    - $z^3 - 3z$
    - $$\frac{r_0}{1+\epsilon\cos b\theta}$$
    - $\cos\left(\pi\frac{p}{q}\right)$
    - $\frac{\cos x^k}{x}$
    - Elliptic integrals and Jacobi elliptic functions
    - The complex exponential $\exp(z)$
    - $\exp\left(\frac{z+1}{z-1}\right)$
    - e^{-x^2}
    - [Only sum terms that are multiples of N in $\mathbb{C}[[z](https://rupadarshiray.github.io/notes/YchnxQ9SF1RsuKjBntMv8a.$]]
    - $\int\frac{1}{x^2+bx+c}$
    - $\frac{\exp(z)}{1-z}$
    - $\sin x$
    - $\sin(x\sin(x))$
    - $\frac{\sin x}{x}$
    - stock.sin x by xk
    - $\sin x^2$
    - $\sqrt{z}$
    - $\sqrt{z^2-1}$
    - Theta functions
    - Weierstrass elliptic function $\wp$
    - Weierstrass' primary factors
    - $x^2\sin\frac{1}{x}$
    - $x+x^2\sin\frac{1}{x}$

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1. why? ↩
2. define this function? ↩