ⓘ Info

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Cylindrical coordinates on S^2

The *cylindrical coordinates*

$$\begin{aligned} S^1 \times (-1,1) &\to S^2 \setminus \{N, S\} \\ (\theta, z) &\mapsto \begin{bmatrix} \sqrt{1-z^2}\cos\theta \\ \sqrt{1-z^2}\sin\theta \\ z \end{bmatrix} \end{aligned}$$

defines a *polar* coordinate chart on S^2 minus the north and south poles.

- The coordinate basis are
    - $$\frac{\partial}{\partial\theta} = \begin{bmatrix} -\sqrt{1-z^2}\sin\theta \\ \sqrt{1-z^2}\cos\theta \\ 0 \end{bmatrix}$$
    - $$\frac{\partial}{\partial z} = \begin{bmatrix} -\frac{1}{\sqrt{1-z^2}} z\cos\theta \\ -\frac{1}{\sqrt{1-z^2}} z\sin\theta \\ 1 \end{bmatrix}$$
- As

  $$\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial z} \right\rangle = z\sin\theta\cos\theta - z\sin\theta\cos\theta = 0$$

  we have a diagonal **metric**

  $$g_{S^2} = \begin{bmatrix} 1-z^2 & \\ & \frac{1}{1-z^2} \end{bmatrix}$$

    - As

      $$\det \begin{bmatrix} 1-z^2 & \\ & \frac{1}{1-z^2} \end{bmatrix} = 1$$

      we have the **volume (area) form**

      $$\lambda_g = \mathrm{d}\theta \wedge \mathrm{d}z$$

This makes the diffeomorphism from cylinder to sphere minus the poles

$$S^1 \times (-1,1) \to S^2 \setminus \{N, S\}$$

a *volume (area) preserving* diffeomorphism.

- $$\det g_{S^2} = \left\| \frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z} \right\| = 1$$
- The **complex structure**

  $$J_p X = \hat{p} \times X$$

  and

  $$\hat{p} = \frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z}$$

  so

  $$\begin{aligned} J_p X &= \hat{p} \times X \\ &= \left(\frac{\partial}{\partial\theta} \times \frac{\partial}{\partial z} \right) \times X \\ &= -\left\langle X, \frac{\partial}{\partial z} \right\rangle \frac{\partial}{\partial\theta} + \left\langle X, \frac{\partial}{\partial\theta} \right\rangle \frac{\partial}{\partial z} \end{aligned}$$

- **Gradient vector fields** looks like

  $$\begin{aligned} & g(\mathrm{grad}(H), -) = \mathrm{d}H \\ \implies & \mathrm{grad}(H)^\intercal \begin{bmatrix} 1-z^2 & \\ & \frac{1}{1-z^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial\theta}, \frac{\partial H}{\partial z} \end{bmatrix} \\ \implies & \mathrm{grad}(H)^\intercal = \begin{bmatrix} \frac{1}{1-z^2}\frac{\partial H}{\partial\theta}, (1-z^2)\frac{\partial H}{\partial z} \end{bmatrix} \end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stretchy
    - S^2 and $\mathbb{C}\boldsymbol{P}^1$
        - Cylindrical coordinates on S^2

And it has 10 siblings.

- stretchy
    - S^2 and $\mathbb{C}\boldsymbol{P}^1$
        - Almost complex structure on S^2 induced from spherical metric and its area form
        - Borsuk-Ulam theorem
        - Cylindrical coordinates on S^2
        - Holomorphic vector bundles on $\mathbb{C}\boldsymbol{P}^1$