Discrete subgroups of Lie groups

Proposition 21.28. *Every discrete subgroup of a Lie group is a closed Lie subgroup of dimension zero.*

Proof. Let G be a Lie group and $\Gamma \subseteq G$ be a discrete subgroup. With the subspace topology, Γ is a countable discrete space and thus a zero-dimensional Lie group. By the closed subgroup theorem, Γ is a closed Lie subgroup of G if and only if it is a closed subset. A discrete subset is closed if and only if it has no limit points in G, so assume for the sake of contradiction that g is a limit point of Γ in G. By discreteness, there is a neighborhood U of e in G such that $U \cap \Gamma = \{e\}$, and then Problem 7-6 shows that there is a smaller neighborhood V of e such that $g_1 g_2^{-1} \in U$ whenever $g_1, g_2 \in V$. Then $Vg = \{hg : h \in V\}$ is a neighborhood of g, and because G is Hausdorff, Vg contains infinitely many points of Γ. Let γ_1, γ_2 be two distinct points in $\Gamma \cap Vg$. Then $\gamma_1 g^{-1}, \gamma_2 g^{-1} \in V$, which implies

$$\gamma_1 \gamma_2^{-1} = (\gamma_1 g^{-1})(\gamma_2 g^{-1})^{-1} \in U.$$

Since $U \cap \Gamma = \{e\}$, this implies $\gamma_1 \gamma_2^{-1} = e$, so $\gamma_1 = \gamma_2$, contradicting our assumption that γ_1 and γ_2 are distinct. ☐
