Info

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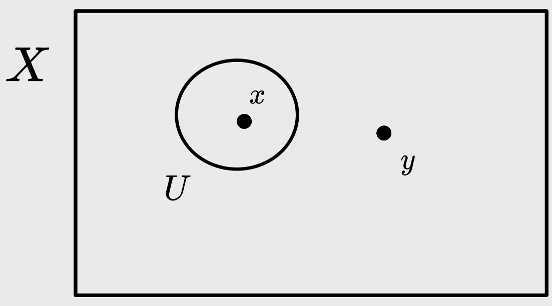
Types of topological spaces

- **Definition.** T_0 topological space

  A topological space is said to be T_0 if for any pair $x, y \in X$, $\exists U \in \mathcal{T}$ such that U contains one of them and not the other.

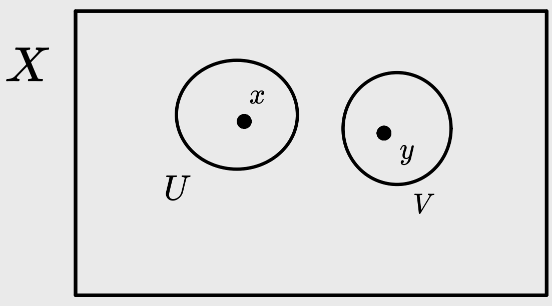
- **Definition.** T_1 topological space

  A topological space is said to be T_1 is for any pair $x, y \in X$, $\exists U, V \in \mathcal{T}$ such that U contains x and not y and V contains y but not x.

- **Definition.** T_2 topological space or Hausdorff space

  A topological space $(X, \mathfrak{T})$ is said to be T_2 **or Hausdorff** if *we can "house off" every pair of points using disjoint open sets*

  $$\begin{aligned}\forall x, y \in X \exists U, V \in \mathfrak{T}\\ \text{st } x \in U, y \in V\\ \text{and } U \cap V = \emptyset\end{aligned}$$

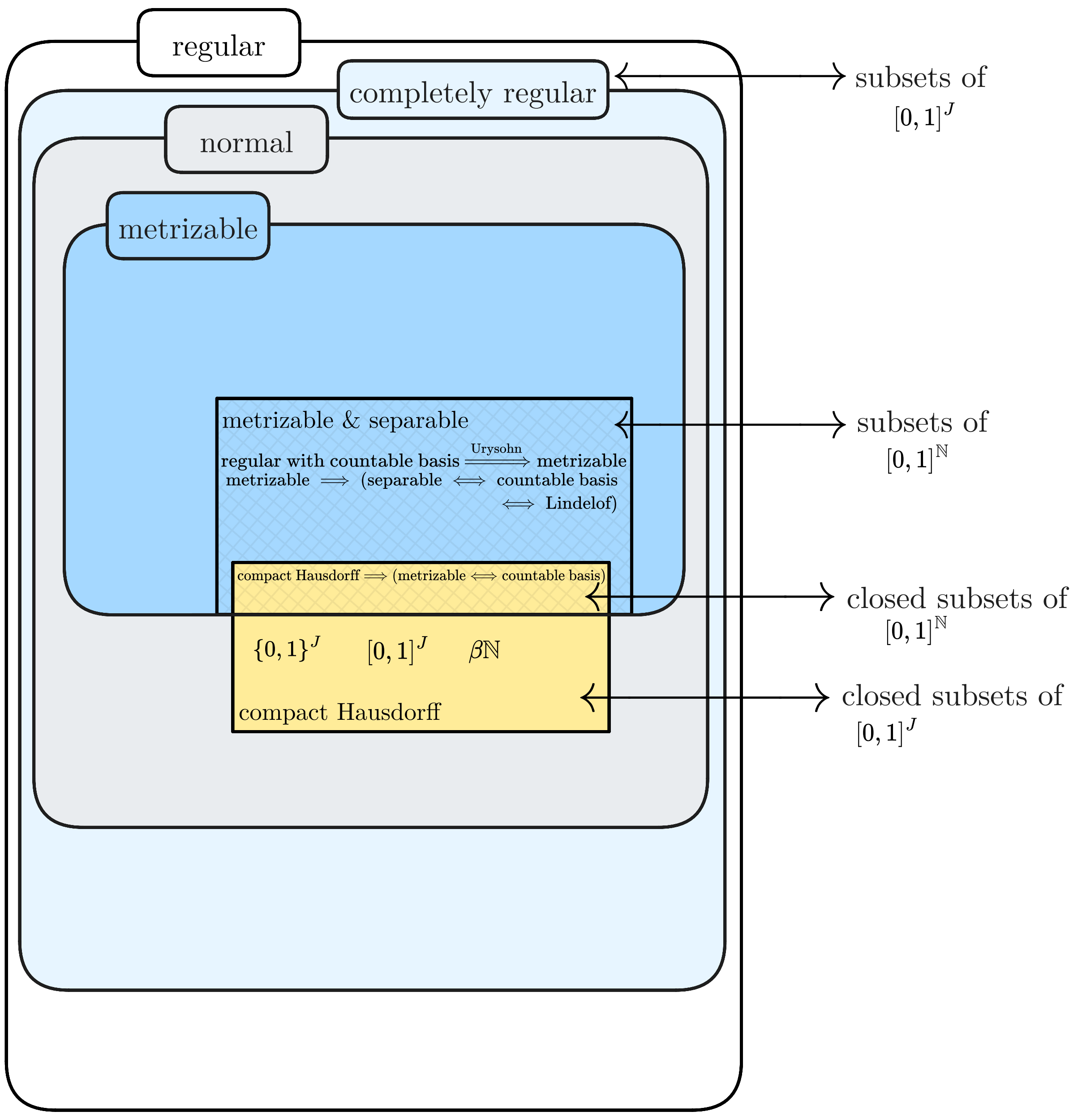
  

- T3 (regular)
- completely regular
- T4 (normal)
- T5
- ...
- **Definition.** A topological space $(X, \mathcal{T})$ is said to be **first countable** if every point in X has a *countable local basis.*
- **Definition.** A topological space $(X, \mathcal{T})$ is called **second countable** or **2-countable** if there is a countable basis on X that generates $\mathcal{T}$.
- **Definition.** A topological space $(X, \mathcal{T})$ is called **separable** if there is a countable, dense subset of X.
- Compact
- Connected
- Path connected
  - Locally path connected
- Simply connected
  - Semi-locally simply connected
  - Locally simply connected
- **Definition.** A Hausdorff space is **locally compact** if every point has a open neighborhood whose closure is compact.

	1-countable	**2-countable**	**countable dense subset (separable)**	**every open cover has a countable subcover (Lindelöf)**
1-countable		2-countable $\implies$ 1-countable		
2-countable				
countable basis			countable basis $\implies$ has a countable dense subset	countable basis $\implies$ Lindelöf

- metrizable $\implies$ normal
  normal $\implies$ regular
  regular $\implies$ Hausdorff

Hausdorff topological spaces



Proposition: regular, countable basis $\implies$ normal

- compact Hausdorff $\implies$ normal

- well-order topology $\implies$ normal

- regular, Lindelöf $\implies$ normal

- **(Urysohn lemma)** Let $A, B \subseteq X$ be disjoint closed subsets of normal space X. Then there exists a continuous map

$$\begin{aligned} f : X &\to [a,b] \\ A &\mapsto a \\ B &\mapsto b \end{aligned}$$

- Let X be a *normal space*.
  - *Choose* $p, q \in X$, because singletons closed subsets we have a *Urysohn function*

$$f : X \to [0,1], f(p) = 0, f(q) = 1$$

    that separates these points.
    - If X is connected and not singleton then we have a surjective map from X to $[0,1]$, so X is uncountable.

- **(Urysohn metrization theorem)** regular space with a countable basis $\implies$ homeomorphic to a subset of $[0,1]^{\mathbb{N}}$ $\implies$ metrizable

- metrizable $\implies$ (separable $\iff$ second countable)

- compact metric space $\implies$ second countable, separable Hausdorff ›

Corollary: Since

- compact metric space $\implies$ second countable, separable Hausdorff ›

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1. https://planetmath.org/acompactmetricspaceissecondcountable ↩

and

> **(Urysohn metrization theorem)** regular space with a countable basis $\implies$ homeomorphic to a subset of $[0,1]^{\mathbb{N}}$ $\implies$ metrizable

This implies: compact Hausdorff space $\implies$ (metrizable $\iff$ it is second-countable).

Definition. A topological space of **σ-compact** if its a countable union of compact subsets.

locally compact, second countable Hausdorff $\implies$ σ-compact

separable, locally compact, metrizable $\implies$ σ-compact

σ-compact metrizable $\implies$ separable

Corollary:

> separable, locally compact, metrizable $\implies$ σ-compact

and

> σ-compact metrizable $\implies$ separable

implies compact metrizable $\implies$ (separable $\iff$ σ-compact)

locally compact metric space $\implies$ (admits a proper/Heine-Borel metric
$\iff$ second countable
$\iff$ σ-compact
$\iff$ Lindelof)

compactification

Definition. Compactification of a space X

A **compactification** of a space X is a **compact Hausdorff** space Y with

$$\iota : X \hookrightarrow Y, \overline{\iota(X)} = Y$$

If X has a compactification, then X must be completely regular.

Current note has 9 direct children and 10 total descendants.

- structures based on sets
  - Topological spaces
    - Types of topological spaces
      - T_0 topological space
      - T_1 topological space
      - T_2 topological space or Hausdorff space
      - Compact Hausdorff spaces
        - f R
          - reconstructing compact Hausdorff X from $\mathcal{C}(X, \mathbb{R})$
      - First countable topological space
      - Second countable topological spaces
      - Compact, metrizable, totally disconnected perfect topological spaces
      - CW complexes
      - Separable topological spaces

And it has 29 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
    - Discrete subgroup acting on coset space
    - Basis and subbasis for a topology
    - Principal G-bundles over topological spaces

$$G \curvearrowright P \to X$$

    - Vector bundles over topological spaces
    - Closure of a subset of topological space
    - Locally compact and 2-countable Hausdorff topological spaces are paracompact with countable refinement
    - Covering dimension of topological spaces
    - $\mathrm{EndTop}(X, Y)$
    - Eilenberg–Steenrod functors

$$(H_n : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Ab}, \partial_n)$$

    - Local fields
    - Local fields of characteristic 0
    - Topological groups
    - Continuous group action on a topological space
    - Intersection of open, dense subsets of a topological space
    - Local topological spaces
    - Covering space of a topological space
    - Borel measures on a topological space
    - Convergence for nets in a topological space
    - Loops and a open cover of a topological space
    - Paths in a topological space
    - Product of topological spaces

- Ringed topological spaces
- Sheaf on open sets of a topological space
- Closed subsets of a topological space
- Discrete subsets of topological spaces
- Types of topological spaces
- Locally compact Hausdorff spaces
- Topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$

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1. https://planetmath.org/acompactmetricspaceissecondcountable ↩
2. https://math.stackexchange.com/questions/57348/hausdorff-locally-compact-and-second-countable-is-sigma-compact ↩