Info

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Displacement function of an isometry

Definition. Displacement function of an isometry

Let X be a metric space and γ be an isometry of X. The **displacement function** of γ is

$$\begin{aligned} d_\gamma : X &\to \mathbb{R}_{\geq 0} \\ x &\mapsto d_\gamma(x) := d(\gamma(x), x) \end{aligned}$$

The **translation length** of γ is the number

$$|\gamma| := \inf \{d_\gamma(x) | x \in X\} \in \mathbb{R}_{\geq 0}$$

The set of points where d_γ attains this infimum is

$$\operatorname{Min}(\gamma) := d_\gamma^{-1} |\gamma|$$

An isometry γ is called

- **semi-simple** if $\operatorname{Min}(\gamma)$ is non-empty.
  - **elliptic** if γ has a fixed point (d_γ attains zero as minimum)
  - **hyperbolic** if d_γ attains a strictly positive minimum
- **parabolic** if $\operatorname{Min}(\gamma)$ is empty

If $\Gamma \to \operatorname{Isom}(X)$ acts by isometries then

$$\operatorname{Min}(\Gamma) := \bigcap_{\gamma \in \Gamma} \operatorname{Min}(\gamma)$$

and the action is called *semi-simple* if all of its elements are semi-simple.

> **Proposition:**
>
> **6.2 Proposition.** *Let X be a metric space and let γ be an isometry of X. Let Γ be a group acting by isometries on X.*
>
> (1) $\operatorname{Min}(\gamma)$ *is γ-invariant and* $\operatorname{Min}(\Gamma)$ *is Γ-invariant.*
>
> (2) *If α is an isometry of X, then $|\gamma| = |\alpha\gamma\alpha^{-1}|$, and $\operatorname{Min}(\alpha\gamma\alpha^{-1}) = \alpha.\operatorname{Min}(\gamma)$. In particular, if α commutes with γ, then it leaves $\operatorname{Min}(\gamma)$ invariant. If N is a normal subgroup of Γ, then $\operatorname{Min}(N)$ is Γ-invariant.*
>
> (3) *If X is a* CAT(0) *space, then the displacement function d_γ is convex, and hence $\operatorname{Min}(\gamma)$ is a closed convex set.*
>
> (4) *If $C \subset X$ is non-empty, complete, convex, and γ-invariant, then $|\gamma| = |\gamma|_C|$ and γ is semi-simple if and only if $\gamma|_C$ is semi-simple. Thus $\operatorname{Min}(\gamma)$ is non-empty if and only if $C \cap \operatorname{Min}(\gamma)$ is non-empty.*

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    - Metric space isometries
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And it has 2 siblings.

- structures based on sets
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