ⓘ Info

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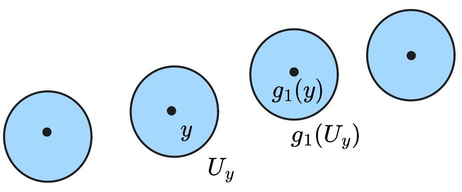
# Covering space action

**Definition.** Covering space action

Let $G$ be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space $Y$ is a **covering space action** if **(orbits are evenly covered)** each $y \in Y$ has a neighborhood $U_y$ such that

$$\begin{aligned} &\{g(U_y) | g \in \mathrm{im}\ G\} \text{ is a pairwise disjoint collection} \\ \iff &g_1, g_2 \in \mathrm{im}\ G, g_1(U_y) \cap g_2(U_y) \neq \emptyset \implies g_1 = g_2 \\ \iff &g \in \mathrm{im}\ G, U_y \cap g(U_y) \neq \emptyset \implies g = \mathsf{Id}_Y \\ \iff &|\{g \in \mathrm{im}\ G | U_y \cap g(U_y) \neq \emptyset\}| = 1 \end{aligned}$$




- covering action $\iff$ the action by $\mathrm{im}\ G$ is **free** and the action is **stably discontinuous** [1]
- On a Hausdorff space, by

  Let $G$ be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a **Hausdorff** topological space $Y$ is **finitely discontinuous**

  $$\iff$$

  $\mathrm{im}\ G$ has **finite stabilizers** and the action is **stably discontinuous**. Moreover, we can take the evenly covered set $U_y$ for any $y \in Y$ to be $(\mathrm{im}\ G)_y$-stable.

  covering $\iff$ the action by $\mathrm{im}\ G$ is **free** and the action is **finitely discontinuous**

  **Definition.** Finitely discontinuous actions on topological spaces

  Let $G$ be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space $Y$ is **finitely discontinuous** if every point $y \in Y$ has a neighborhood $U_y$ such that $U \cap g(U)$ is nonempty for only finitely many $g \in \mathrm{im}\ G$, that is,

  $$|\{g \in \mathrm{im}\ G | U_y \cap g(U_y) \neq \emptyset\}| < \infty$$

So a faithful *covering space action*

$$G \hookrightarrow \mathrm{Homeo}(Y)$$

in particular implies

- the orbits

  $$G\{y\}$$

  are discrete
- stabilizers of a point fix everything

  $$G_x = \{i_G\}$$

## the quotient map is a normal covering

**Proposition:** Let $G \to \mathrm{Homeo}(Y)$ be a covering space action. Then

$$p : Y \to {}_G\backslash^Y$$

is a normal covering space.

For any $y \in Y$ let $U_y$ be the onb of $y$ such that

$$\{g(U_y) | g \in \mathrm{im}\ G\}$$

is a pairwise disjoint collection.

- Thus

  $$p^{-1}(p(U_y)) = \coprod_{g \in \mathrm{im}\ G} g(U_y)$$

- Also because $p$ is a open continuous map

  $$p : g(U_y) \to p(U_y)$$

  is a homeomorphism for each $g \in G$.
- Thus $p$ is a normal covering space.

Group action which is not a *covering action* but the quotient map is a normal covering space.

Consider

$$\begin{array}{ccc} \mathbb{Z} & \curvearrowright & \mathbb{R} \coprod \mathbb{R} \\ & & \big\downarrow{\scriptstyle \mathsf{Id}_{\mathbb{R}} \coprod \exp} \\ & & \mathbb{R} \coprod S^1 \end{array}$$

where $\mathbb{Z}$ acts by $\mathsf{Id}$ by the first component and translations on the second component of

$\mathbb{R} \coprod \mathbb{R}$

$$(x,1),(y,2) \overset{n}{\mapsto} (x,1),(y+n,2)$$

This action is not *free* as stabilizer of $(x,1)$ is the whole group $\mathbb{Z}$.
Thus this action is **not** a covering action.
However, the map $p$ is a normal covering map with $\mathbb{Z}$ being the whole covering automorphism group.

**Theorem 12.14 (Covering Space Quotient Theorem).** *Let $E$ be a connected, locally path-connected space, and suppose we are given an effective action of a group $\Gamma$ on $E$ by homeomorphisms. Then the quotient map $q\colon E \to E/\Gamma$ is a covering map if and only if the action is a covering space action. In this case, $q$ is a normal covering map, and $\mathrm{Aut}_q(E) = \Gamma$, considered as a group of homeomorphisms of $E$.*

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**Proposition:** Let

$$G \curvearrowright Y$$

is a faithful? **covering space action**, then

- if $Y$ is path connected

$$G = \mathrm{AutTop}\left(Y \to {}^{Y}\!/_{G}\right)$$

- if $Y$ is path connected and locally path connected

$$\mathrm{im}\ G \cong \frac{\pi_1\left({}^{Y}\!/_{G}\right)}{\pi_*\pi_1(Y)}$$

## on a topological manifold

**Definition.** coHausdorff actions

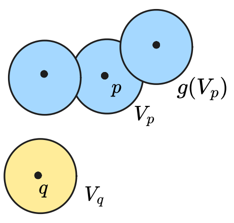
Let $G$ be a group and

$$G \to \mathrm{Homeo}(Y)$$

be an *action by homeomorphisms* on a topological space $Y$. The action is called **coHausdorff** if it satisfies *either* of

- ${}_{\Gamma}\backslash^{X}$ is Hausdorff
- if $p,q \in X$ are not in same $\Gamma$-orbit then there exists neighborhoods $V_p$ of $p$ and $V_q$ of $q$ such that

$$\forall g \in \Gamma, g(V_p) \cap V_q = \emptyset$$




**Proposition:** Let a **discrete** Lie group $\Gamma$ acts **continuously and freely** on a topological manifold $X$. Then the action is **proper** $\iff$ it is a covering space action and coHausdorff $\iff$ it satisfies

**Definition.** discontinuosly coHausdorff actions on topological spaces

Let $G$ be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space $Y$ is **discontinuosly coHausdorff** if it satisfies *any* of

- for each $x_1, x_2 \in X$ there exists respective nb $U_1, U_2 \subseteq X$ such that for every $\gamma \in \Gamma$

$$\gamma(U_1) \cap U_2 \neq \emptyset \iff \gamma(x_1) = x_2$$

- stably discontinuous and

**Definition.** coHausdorff actions

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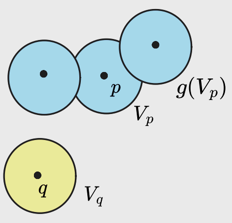


- free and proper $\implies$
  - quotient is Hausdorff
  - ...
- Let the action be coHausdorff and covering.
  - covering $\implies$ free

- Let $\{g_i\} \subseteq \Gamma, \{p_i\} \subseteq X$ such that

$$p_i \xrightarrow{i\to\infty} p, g_i(p_i) \xrightarrow{i\to\infty} p'$$

- If $p, p'$ are in different orbits, it contradicts coHausdorff property.
- So there exists $g \in \Gamma$ such that $g(p) = p'$. This implies

$$g^{-1}g_i(p) \xrightarrow{i\to\infty} p$$

- By covering action property,

$$g^{-1}g_i = \mathsf{Id}$$

for large enough $i$.
- Thus

$$g = g_i$$

which certainly converges. So the action is proper.

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
      - Covering space action

And it has 3 siblings.

- structures based on sets
  - Topological spaces
    - Group action on topological spaces
      - Covering space action
      - Discontinuous group actions on topological spaces
      - Proper actions by topological groups on topological spaces

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1. **Definition.** Stably discontinuous actions on topological spaces

Let $G$ be a group. An *action by homeomorphisms* $G \to \mathrm{Homeo}(Y)$ on a topological space $Y$ is **stably discontinuous** if every point $y \in Y$ has a neighborhood $U_y$ such that

$$\{g \in \mathrm{im}\ G | U_y \cap g(U_y) \neq \emptyset\} = (\mathrm{im}\ G)_y$$

↩