ⓘ Info

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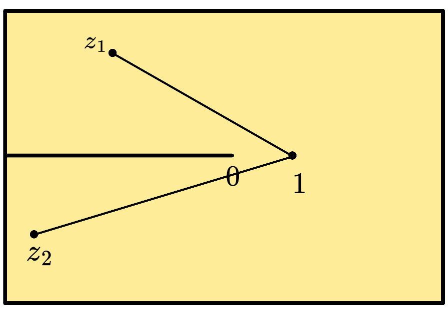
The complex logarithm

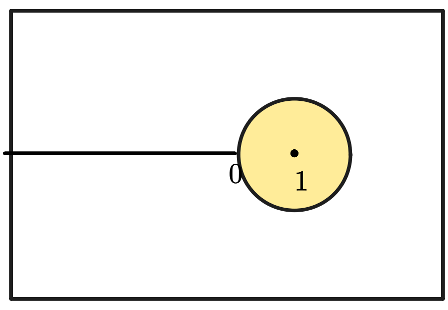
$$-\log(1-z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots$$
$$\log\frac{1}{1-z} = \sum_{n\geq 1}\frac{z^n}{n}$$

complex logarithm on the split plane

Definition. The complex logarithm

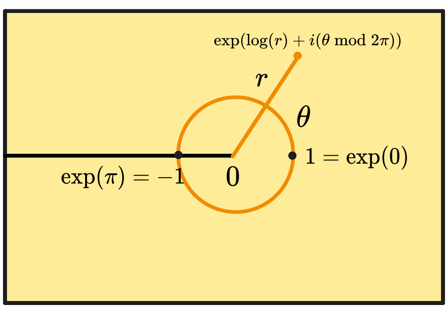
$$\mathbb{C}\setminus(-\infty,0] \to \mathbb{C}$$
$$\log(z) := \int_{1\mapsto z}\frac{dw}{w}$$





- $$\log(1-z) = \sum_{n\geq 0}\frac{z^{n+1}}{n+1} \quad \text{on } |z|<1$$
    - $$\log(z) = \log(1-(1-z)) = \sum_{n\geq 0}\frac{(1-z)^{n+1}}{n+1} \quad \text{on } |1-z|<1$$
- By definition

$$\frac{\partial}{\partial z}\log(z) = \frac{1}{z}$$

> **Proposition:**
>
> $$\log(\exp(z)) = z$$
>
> therefore $\log(re^{i\theta}) = \log(r) + i\theta$
>
> 
>
>
> and
>
> $$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

- $$\frac{\partial}{\partial z}\log(\exp(z)) = \frac{1}{\exp(z)}\exp(z) = 1$$

the Riemann surface of $\log$ is $U(\mathbb{C}^\times)$

Definition.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, phantom, "\curvearrowright"] & (U(\mathbb{C}^\times),+) \arrow[d, "\pi"] \\ & (\mathbb{C}^\times,\times) \end{tikzcd}$$

Consider

$$\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$$

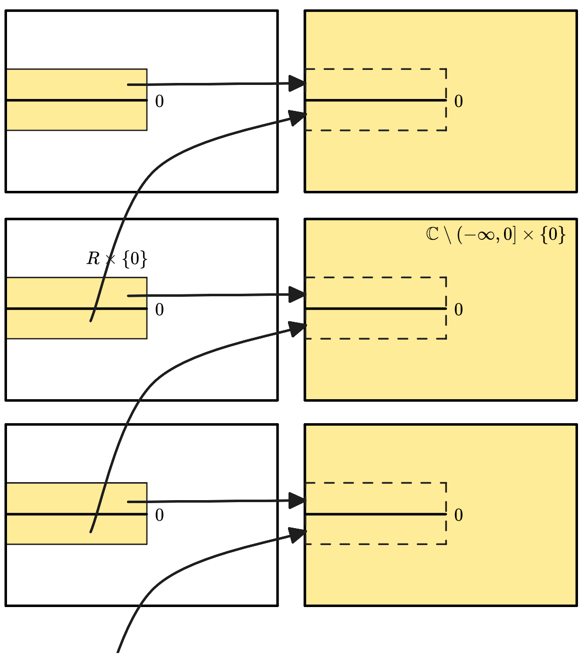
and take the open subset $\mathbb{C}\setminus(-\infty,0]$ and the open rectangle $R := (-\infty,0)\times(-1,1)$ and take $\mathbb{Z}$ many copies

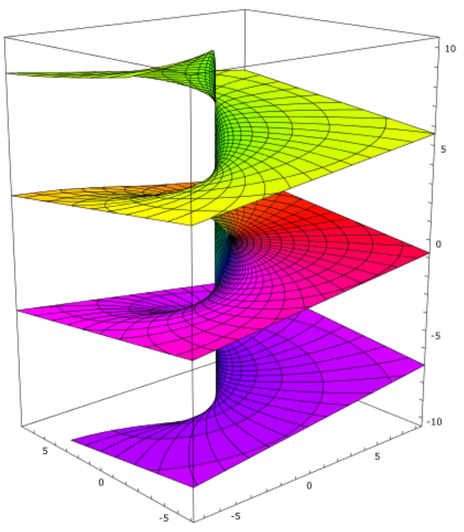
$$\mathbb{C}\setminus(-\infty,0]\times\mathbb{Z}$$
$$R\times\mathbb{Z}$$

and consider the gluing functions

$$g_k : (R\setminus(-\infty,0))\times\{k\} \to \mathbb{C}\setminus(-\infty,0]\times\mathbb{Z}$$
$$(z,k) \mapsto \begin{cases}(z,k) & \Im z > 0\\ (z,k+1) & \Im z < 0\end{cases}$$

for every $k\in\mathbb{Z}$





and glue them together using the gluing functions

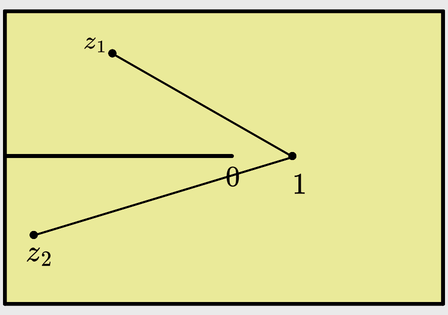
$$\begin{aligned} U(\mathbb{C}^\times) := \frac{\mathbb{C} \setminus (-\infty, 0] \times \mathbb{Z} \coprod R \times \mathbb{Z}}{g_k, \forall k \in \mathbb{Z}} &\xrightarrow{\pi} \mathbb{C}^\times \\ [z,k] &\mapsto z \end{aligned}$$

- We have the branches of the log function on the $\mathbb{Z}$-many sheets
  - $$\begin{aligned} \mathbb{C} \setminus (-\infty, 0] \times \mathbb{Z} &\to \mathbb{C} \\ (z,k) &\mapsto \log(z) + 2\pi i k \end{aligned}$$

    where log

    Definition. The complex logarithm

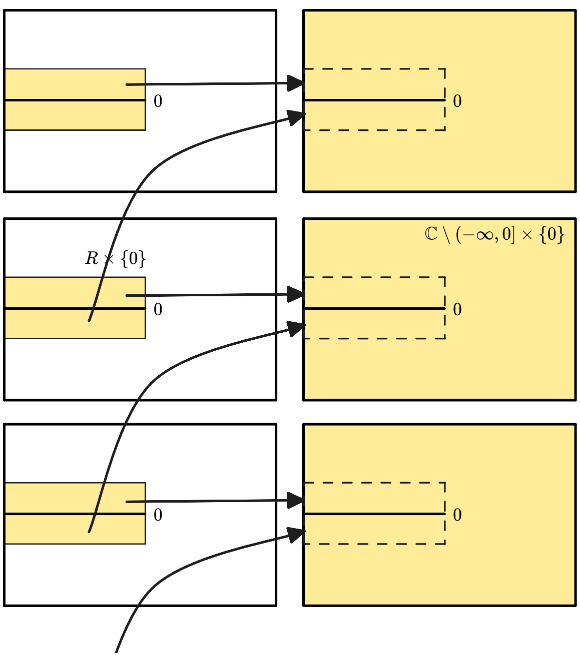
    $$\begin{aligned} \mathbb{C} \setminus (-\infty, 0] &\to \mathbb{C} \\ \log(z) &:= \int_{1 \mapsto z} \frac{\mathrm{d}w}{w} \end{aligned}$$

  - and

    $$\begin{aligned} R \times \mathbb{Z} &\to \mathbb{C} \\ (z,k) &\mapsto \int_{-1 \mapsto z} \frac{\mathrm{d}w}{w} + i\pi + 2\pi i k \end{aligned}$$

  These functions agree with the gluing functions

Thus we have a continuous log on $U(\mathbb{C}^\times)$ which is actually holomorphic

> **Proposition:** The function
>
> $$\log : U(\mathbb{C}^\times) \to \mathbb{C}$$
>
> is holomorphic.

Thus we have

$$\begin{tikzcd} \log : U(\mathbb{C}^\times) \arrow[d, "\pi"] \to \mathbb{C} \\ \mathbb{C} \end{tikzcd}$$

[1]

$\log$ on $U(\mathbb{C}^\times)$ is a biholomorphism to $\mathbb{C}$

> **Proposition:** The holomorphic function
>
> $$\begin{aligned}\log : U(\mathbb{C}^\times) &\to \mathbb{C}\\ (z,\theta) &\mapsto \log(|z|) + i\theta\end{aligned}$$
>
> is a bijection and thus a biholomorphism.

Therefore, we have a commutative diagram

$$\begin{tikzcd} U(\mathbb{C}\setminus\{0\}) \\ (z,\theta) \arrow[rr, bend left, "\log"] \arrow[dr, "\pi"'] & & \mathbb{C} \\ \log(z,\theta) \arrow[ll, bend left] \arrow[dl, "\exp"] \\ & \mathbb{C}\setminus\{0\} \\ \pi(z,\theta) = z = \exp(w) & \end{tikzcd}$$

Riemann surface of $\log$ through a holomorphic 1-form

The 1-form

$$\frac{1}{z}\mathrm{d}z$$

is a closed 1-form on $\mathbb{C}^\times$ but not exact as

$$\int_{S^1} \frac{1}{z}\mathrm{d}z = 2\pi i$$

But when pulled back onto $U(\mathbb{C}^\times)$ the 1-form

$$\frac{1}{z}\mathrm{d}z \text{ on } U(\mathbb{C}^\times)$$

is exact because $U(\mathbb{C}^\times)$ is simply connected.

The function

> **Proposition:** The function
>
> $$\log : U(\mathbb{C}^\times) \to \mathbb{C}$$
>
> is holomorphic.

is precisely its anti-derivative.

Current note has 0 direct children and 0 total descendants.

- squishy
  - pow
    - The complex logarithm

And it has 28 siblings.

- squishy
  - pow
    - Angle function
    - Bessel functions
    - $$\cos\sqrt{z}$$
    - The complex logarithm
    - $\sum_{n\geq 0} z^{n!}$
    - $z^3 - 3z$
    - $$\frac{r_0}{1+\epsilon\cos b\theta}$$
    - $\cos\left(\pi\frac{p}{q}\right)$
    - $\frac{\cos x^k}{x}$
    - Elliptic integrals and Jacobi elliptic functions
    - The complex exponential $\exp(z)$
    - $\exp\left(\frac{z+1}{z-1}\right)$
    - e^{-x^2}
    - [Only sum terms that are multiples of N in $\mathbb{C}[[z](https://rupadarshiray.github.io/notes/YchnxQ9SF1RsuKjBntMv8a.$]]
    - $\int\frac{1}{x^2+bx+c}$
    - $\frac{\exp(z)}{1-z}$
    - $\sin x$
    - $\sin(x\sin(x))$
    - $\frac{\sin x}{x}$
    - stock.sin x by xk
    - $\sin x^2$
    - $\sqrt{z}$
    - $\sqrt{z^2-1}$
    - Theta functions
    - Weierstrass elliptic function $\wp$
    - Weierstrass' primary factors
    - $x^2\sin\frac{1}{x}$
    - $x + x^2\sin\frac{1}{x}$

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1. Elias Wegert - Visual Complex Functions_ An Introduction with Phase Portraits-Birkhäuser Basel (2012), p.317 ↩