Info

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Exponential sum of a discrete subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$

> Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

Intuition

We are interested in the **rate at which orbit tends to** S^{n-1}.

- Any two orbits $\Gamma(a), \Gamma(b)$ are *comparable*: As

> **Proposition:**
> $$d_{\mathbb{R}\boldsymbol{H}^n}(o,x) = \log\left(\frac{1+|x|}{1-|x|}\right)$$

$$\begin{aligned} & d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(b)) \leq d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(a)) + d_{\mathbb{R}\boldsymbol{H}^n}(a,b) \\ \implies & \log\left(\frac{1+|\gamma(b)|}{1-|\gamma(b)|}\right) \leq \log\left(\frac{1+|\gamma(a)|}{1-|\gamma(a)|}\right) + d_{\mathbb{R}\boldsymbol{H}^n}(a,b) \\ \implies & \frac{1-|\gamma(a)|}{1-|\gamma(b)|} \leq 2e^{d_{\mathbb{R}\boldsymbol{H}^n}(a,b)} \end{aligned}$$

- A good measure of rate of convergence of $\gamma(a) \xrightarrow{\gamma\in\Gamma} S^{n-1}$ is

$$\alpha \mapsto \sum_{\gamma\in\Gamma}(1-|\gamma(a)|)^\alpha$$

for $\alpha > 0$. This convergence is independent of a by above.

- However, it is more natural to consider

$$\alpha \mapsto \sum_{\gamma\in\Gamma}\exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o)))$$

and because

$$\exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o))) = \left(\frac{1-|\gamma(o)|}{1+|\gamma(o)|}\right)^\alpha \leq (1-|\gamma(o)|)^\alpha$$

the two series convergence or diverge together for fixed $\alpha > 0$.

- Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum

$$\begin{aligned} & \sum_{\gamma\in\Gamma, d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))<R} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o))) \\ &= \int_{(0,R)} \exp(-\alpha t)\mathrm{d}N(t,o,o) \\ &= N(R,o,o)e^{-\alpha R} + \alpha\int_{t\in(0,R)} N(t,o,o)e^{-\alpha t}\mathrm{d}t \end{aligned}$$

> **Proposition:**
> > Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.
>
> Let $\alpha > n-1, x \in X$. Then
> $$\sum_{\gamma\in\Gamma}\exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))) < \infty$$

Consider the partial sum

- Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum

$$\begin{aligned} & \sum_{\gamma\in\Gamma, d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))<R} \exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o))) \\ &= \int_{(0,R)} \exp(-\alpha t)\mathrm{d}N(t,o,o) \\ &= N(R,o,o)e^{-\alpha R} + \alpha\int_{t\in(0,R)} N(t,o,o)e^{-\alpha t}\mathrm{d}t \end{aligned}$$

and by

> > Let
> > $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
> > be a discrete subgroup.
>
> Let $y \in \mathbb{R}\boldsymbol{H}^n$. Then there is a $A_{\Gamma,y} > 0$ such that
> $$\forall x \in \mathbb{R}\boldsymbol{H}^n, N(r,x,y) \leq A_{\Gamma,y}e^{r(n-1)}$$

we have

$$\begin{aligned} \alpha > n-1 \implies & \int_{t\in(0,R)} A_{\Gamma,o}e^{t(n-1)-\alpha t}\mathrm{d}t \\ =& A_{\Gamma,o}\frac{1}{n-1-\alpha}(e^{R(n-1-\alpha)}-1) \\ & \xrightarrow{R\to\infty} < \infty \end{aligned}$$

Definition. Critical exponent of discrete subgroup

> Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

$$\delta_\Gamma := \inf\left\{\alpha \middle| \sum_{\gamma\in\Gamma} \exp(-\alpha d(x,\gamma(x))) < \infty\right\}$$
$$\in [0, n-1]$$

Proposition:

> Let $\Gamma \leq \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$ be a discrete subgroup.

If Γ has finite covolume, then

$$\sum_{\gamma\in\Gamma} \exp(-\alpha d(x,\gamma(x)))$$

diverges for $\alpha = n-1$, so $\delta_\Gamma = n-1$.

Consider the partial sum

> - Let $o \in \mathbb{R}\boldsymbol{H}^n$ be fixed. We consider the partial sum
>
> $$\begin{aligned}\sum_{\gamma\in\Gamma, d_{\mathbb{R}\boldsymbol{H}^n}(x,\gamma(x))<R} &\exp(-\alpha d_{\mathbb{R}\boldsymbol{H}^n}(o,\gamma(o))) \\ &= \int_{(0,R)} \exp(-\alpha t)\mathrm{d}N(t,o,o) \\ &= N(R,o,o)e^{-\alpha R} + \alpha\int_{t\in(0,R)} N(t,o,o)e^{-\alpha t}\mathrm{d}t\end{aligned}$$

Now by

> - Thus for $r > R$
>
> $$N(r,o,y) \geq A' e^{(n-1)r}$$

we have

$$\begin{aligned}\sum_{\gamma\in\Gamma} \exp(-(n-1)d(x,\gamma(x))) &\geq N(R,o,o)e^{(n-1)R} \\ &+ (n-1)A'\int_{t\in(0,R)} \cancelto{1}{e^{(n-1)t}e^{-(n-1)t}}\mathrm{d}t \\ &\geq N(R,o,o)e^{(n-1)R} + (n-1)A'R \\ &\xrightarrow{R\to\infty} \infty\end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Exponential sum of a discrete subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$

And it has 11 siblings.

- stem
  - O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
      - Convex cocompact discrete subgroups
      - Combination of discrete subgroups of $O^+(1,n)(\mathbb{R})$
      - Exponential sum of a discrete subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$
      - $\mathrm{RepDF}(F_g, O^+(1,n)(\mathbb{R}))$
      - Measure homology on a hyperbolic manifold
      - Discrete subgroups with limit set a discontinuum
      - Limit sets of discrete subgroups
      - Discrete subgroups with limit set a $k-1$-sphere
      - Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$
      - Rigidity of $\mathbb{R}$-hyperbolic manifolds
      - $\mathrm{RepDF}(\pi_1(T_g^2,\bullet), O^+(1,n)(\mathbb{R}))$