Info

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Klein geometry

Definition. Klein geometry

A **Klein geometry** is a homogeneous manifold

$$G \curvearrowright {}^{G}/_{H}$$

where $H \leq G$ is a closed subgroup of a Lie group G such that G/H is connected.

- The **kernel** of a Klein geometry is

$$\ker\left(G \to \mathsf{AutMan}({}^{G}/_{H})\right) \trianglelefteq G$$

- The **associated pair of Lie algebras** is $\mathfrak{h} \leq \mathfrak{g}$ where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively.
- The **associated principle H-bundle** is

$$\begin{tikzcd} G \arrow[d] & \curvearrowleft & H \\ {}^{G}/_{H} & & \end{tikzcd}$$

A Klein geometry is

- **faithful** if its kernel is trivial and **locally faithful** if its kernel is discrete in G
- **geometrically oriented** if G is connected
- **primitive** if the identity component $H_\circ \trianglelefteq H$ is a maximal proper closed connected subgroups of G
- **reductive** if there is an Ad_H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$

The **associated faithful Klein geometry** of a Klein geometry with kernel K is

$$G/K \curvearrowright {}^{G/K}/_{H/K}$$

Definition. Mutants of Klein geometries

Two Klein geometries $H_1 \leq G_1$ and $H_2 \leq G_2$ are **mutants** of each other if there is an isomorphism of Lie groups

$$\varphi : H_1 \cong H_2$$

such that the induced $\mathfrak{D}_i\varphi : \mathfrak{h}_1 \to \mathfrak{h}_2$ extends to a $\mathbb{R}$-linear isomorphism

$$\lambda : \mathfrak{g}_1 \to \mathfrak{g}_2$$

which is a $\mathrm{Ad}_{H_1 \cong H_2}$-module morphism, that is,

$$\lambda(\mathrm{Ad}_h X) = \mathrm{Ad}_{\varphi(h)}(\lambda(X))$$

Definition. Locally Klein geometry

Let

$$G \curvearrowright {}^{G}/_{H}$$

be a homogeneous manifold where $H \leq G$ is a closed subgroup of a Lie group G and $\Gamma \leq G$ is a discrete subgroup that acts

$$\Gamma \curvearrowright {}^{G}/_{H}$$

faithfully as a group of covering transformations such that $\Gamma \backslash {}^{G}/_{H}$ is *connected*. Then the diagram

$$\begin{tikzcd} G & \curvearrowright & {}^{G}/_{H} \arrow[d] \\ & & \Gamma \backslash {}^{G}/_{H} \end{tikzcd}$$

is called a **locally Klein geometry**.

Proposition: Let

$$\begin{tikzcd} G & \curvearrowright & {}^{G}/_{H} \arrow[d] \\ & & \Gamma \backslash {}^{G}/_{H} \end{tikzcd}$$

be a *locally Klein geometry* then

$$\begin{tikzcd} \Gamma \backslash G \arrow[d] & \curvearrowleft & H \\ \Gamma \backslash {}^{G}/_{H} & & \end{tikzcd}$$

is a **principal H-bundle**.

Definition. Klein pair

A **Klien pair** is a pair of Lie algebras

$$\mathfrak{h} \leq \mathfrak{g}$$

(subalgebra)
The **kernel** if such a pair is the largest ideal $\mathfrak{k} \trianglelefteq \mathfrak{g}$ contained in $\mathfrak{h}$.
If the kernel is trivial $\mathfrak{k} = 0$, the pair is **faithful**.
If there is a $\mathfrak{h}$-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ then the pair is **reductive**.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - Homogeneous manifold
      - Klein geometry

And it has 8 siblings.

- structures based on sets
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