Info

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Homogeneous manifold with compact stabilizer

Proposition?: Let

$$G \curvearrowright {}^{G}/_{K}$$

be a homogeneous manifold where $K \leq G$ is a closed subgroup of a Lie group G.

- The action of $G \curvearrowright {}^{G}/_{K}$ is **proper** $\iff$ K is **compact**.
- If K is compact, there is a G-invariant *complete* Riemannian metric g_K on ${}^{G}/_{K}$ such that

$$G \to \operatorname{Isom}\left({}^{G}/_{K}, g_K\right)$$

  has closed image and

$$\dim \frac{G}{\ker} \leq \dim \operatorname{Isom}\left({}^{G}/_{K}, g_K\right) \leq \frac{1}{2}(\dim G - \dim K)(\dim G - \dim K + 1)$$

-

Assume the action is not proper. [1]

- Then there exists sequences $\{p_i\}, \{g_i p_i\} \subset M$ such that no subsequence of $\{g_i\} \subset G$ converge.
- Now pick representatives $\{\gamma_i\} \subset G$ such that $\gamma_i K = p_i$

$$\begin{tikzcd} G \arrow[r, phantom, "\overset{\text{proper}}{\curvearrowright}"] & G \arrow[d, "\text{proper}"] \\ G \arrow[r, phantom, "\curvearrowright"] & {}^{G}/_{K} \end{tikzcd}$$

- Thus $\{\gamma_i\}$ is a subset of a compact set in G, so has a convergent subsequence.
- Similarly, $\{g_i \gamma_i\}$ also has convergent subsequence.
- However, because no subsequence of $\{g_i\} \subset G$ converge, this shows the action $G \curvearrowright G$ is proper. This is a contradiction.

$$\begin{tikzcd} G \times G \arrow[r, "\text{proper}"] \arrow[d] & G \times G \arrow[d, "\mathrm{Id} \times \text{proper}"] \\ G \times {}^{G}/_{K} \arrow[r] & {}^{G}/_{K} \times {}^{G}/_{K} \end{tikzcd}$$

$$(g, h) \mapsto (h, gh)$$

$$(g, hK) \to (hK, ghK)$$

- By

  Let $G \curvearrowright X$ be a **transitive** and smooth Lie group action. Then X admits a G-invariant Riemannian metric $\iff$ for some (hence all) $x \in X$ the image

$$G_x \to GL(T_x X)$$

  has compact closure $\iff$ the image is conjugate to a subgroup of $O(T_x X)$.

  there exists such a g_K and by

  (Myers-Steenrod) Let (M, g) be a Riemannian manifold. Then $\operatorname{Isom}(M, g)$ with the compact-open topology

  Definition. Compact-open topology on Y^X

  Let X be a topological space and (Y, d) be a metric space. Let $K \subseteq X$ be compact and $\epsilon > 0$ and consider the subsets

$$B_\epsilon(f, K) := \left\{ g \in Y^X \middle| \sup_{x \in K} d(f(x), g(x)) < \epsilon \right\}$$

  The topology on Y^X generated by these subsets is the **compact-open** topology $\mathfrak{T}_{\text{cpt}}$.

  is a Lie group such that the action

$$\operatorname{Isom}(M, g) \curvearrowright M$$

  is **smooth** and **proper**.
  If M is connected, we have

$$\dim \operatorname{Isom}(M, g) \leq \frac{1}{2} \dim M(\dim M + 1)$$

  with equality $\iff$ M is isometric to $\mathbb{R}^n$ or S^n or $\mathbb{R}\boldsymbol{P}^n$ or $\mathbb{R}\boldsymbol{H}^n$ upto scaling.
  If (M, g) is complete and connected?, the action of $G \leq \operatorname{Isom}(M, g)$ is **proper** $\iff$ G is closed in $\operatorname{Isom}(M, g)$.

  we have the image closed.

Question

Let G be connected. Then when is

$$G \to \operatorname{Isom}\left({}^{G}/_{K}, g_K\right)_\circ$$

surjective?

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - Homogeneous manifold
      - Homogeneous manifold with compact stabilizer

And it has 8 siblings.

- structures based on sets
  - manifolds
    - Homogeneous manifold
      - Analytic geometries
      - Invariant Riemannian metrics on homogeneous manifolds
      - Klein geometry
      - Nilmanifolds
      - Reductive homogeneous spaces
      - Naturally reductive homogeneous spaces
      - Homogeneous manifold with compact stabilizer
      - Thurston geometries

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1. lie groups - proper action on homogeneous space - Mathematics Stack Exchange ↩