Info

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Generators of a Lie group action

derivative

Definition. Derivative of a smooth Lie group action on a smooth manifold

Let $\Phi : G \curvearrowright M$ is a Smooth Lie group action by G on M then we have

$$\begin{aligned} \mathrm{d}\Phi : \mathfrak{g} &\to \mathrm{Vec}(M) \\ v &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi^{\exp(tv)} \end{aligned}$$

We think of it as the "derivative"

$$\mathrm{d}\Phi : \mathfrak{g} \to \mathrm{Vec}(M)$$

$$\mathrm{d}\Phi(v)_p = \mathrm{d}_{i_G}\Phi(-,p)(v)$$

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is a Lie algebra homomorphism

Hence, the action

$$\Phi^{\exp tv_1}, \Phi^{\exp tv_2}$$

commute if and only if $[v_1, v_2] = 0$

lifting Lie algebra action to Lie group action

Let M be a smooth manifold and G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Suppose

$$\theta : \mathfrak{g} \to \mathrm{Vec}_{\mathsf{cpl}}(M)$$

is a complete $\mathfrak{g}$-action on M. Then there is a **unique smooth G-action** on M whose infinitesimal generator is θ.

Bug

We need to construct a map

$$\Phi : G \times M \to M$$

but we cannot just say

$$\Phi^{\exp(tX)} := \exp(t\theta(X))$$

(for $X \in \mathfrak{g}$) because the exponential $\exp : \mathfrak{g} \to G$ is **not surjective** in general, so not all elements of g look like $\exp(tX)$ for some X.

- For each $X \in \mathfrak{g}$ define a smooth vector field $\tilde{X}$ on $G \times M$ by

$$\begin{aligned} \tilde{X} : G \times M &\to T(G \times M) \cong TG \times TM \\ \tilde{X}_{(g,p)} &= (X_g, \theta(X)_p) \end{aligned}$$

- This gives a map

$$\mathsf{Id}_{\mathfrak{g}} \times \theta : \mathfrak{g} \to \mathrm{Vec}(G \times M) = \mathrm{Vec}(G) \oplus \mathrm{Vec}(M)$$

  which is a Lie algebra homomorphism.
- Let D be the smooth distribution defined on $G \times M$ by vector fields in the image of this map.
- For the left G action on $G \times M$ by

$$(g_1, p) \overset{\psi_g}{\longmapsto} (gg_1, p)$$

  so $\psi_g = L_g \times \mathsf{Id}_M$ we observe

$$\begin{aligned} \mathrm{d}_{(g_1,p)}\psi_g(\underbrace{\tilde{X}_{(g_1,p)}}_{(X_g,\theta(X)_p)}) &= (\mathrm{d}_{g_1}L_g(X_{g_1}), \theta(X)_p) \\ &= (X_{gg_1}, \theta(X)_p) \\ &= \tilde{X}_{gg_1,p} \end{aligned}$$

  > - So D is invariant under ψ_g for each $g \in G$. So ψ_g takes the leaves of the foliation $F(D)$ to leaves of $F(D)$.

- $$\Pi_p := \pi_G\Big|_{F(D)_p} : F(D)_p \to G$$

  where$F(D)_p$ is the leaf containing (i_G, p).

  > - $$\mathrm{d}_{(g,q)}\Pi_p(\tilde{X}_{(g,q)}) = X_g$$

- > It is a covering map. As G is simply connected, it is thus a diffeomorphism.

- For each $p \in M$

$$\theta^{(p)} : G \to M$$
$$\theta^{(p)} := \pi_M \circ \Pi_p^{-1}$$

and the map

$$\Phi : G \times M \to M$$
$$\Phi^g(p) := \theta^{(p)}(g)$$

  - This is equivalent to declaring

$$\Phi^g(p) = q \iff F(D)_{(i_G,p)} = F(D)_{(g,q)}$$

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
        - Generators of a Lie group action

And it has 5 siblings.

- structures based on sets
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    - smooth $\mathbb{R}$-manifold
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        - Smooth action by a compact Lie group
        - Generators of a Lie group action
        - Orbits of smooth Lie group actions
        - Proper Lie group actions
        - invariant Riemannian metric in a proper Lie group action