Info

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Biholomorphic mapping onto disk

(Riemann mapping theorem) Let $\Omega \subset \mathbb{C}$ be a simply connected open, proper subset of $\mathbb{C}$. Then there exists a biholomorphism from Ω onto the unit disk

$$f : \Omega \to D$$

Moreover, given $a \in \Omega$ there is a **unique** biholomorphism

$$\begin{gathered} f : \Omega \to D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$

uniqueness and maximality condition

> **Proposition:** Let $\Omega \subseteq \mathbb{C}$ be a proper, simply connected subset of $\mathbb{C}$. Then
>
> 1. Given $a \in \Omega$ there is at most one biholomorphism
>
> $$\begin{gathered} f : \Omega \cong D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$
>
> 2. There exists an injective holomorphic map
>
> $$f : \Omega \hookrightarrow D$$
>
> 3. Let $f : \Omega \to D$ be injective holomorphic such that $f(a) = 0$. Then the map f attain **maximum** of
>
> $$\begin{aligned} \{h \in \mathscr{O}(\Omega, D) \mid h \text{ is injective }, h(a) = 0\} &\to [0, \infty) \\ h &\mapsto |h'(a)| \end{aligned}$$
>
> $\iff$ f is surjective.

uniqueness

For a give $a \in \Omega$, let there are two biholomorphisms

$$f, g : \Omega \to D$$

then

$$f \circ g^{-1} : D \to D$$

is a automorphism of the disk such that

$$\begin{gathered} f \circ g^{-1}(0) = f(a) = 0 \\ (f \circ g^{-1})'(0) = f'(g^{-1}(0)) \frac{1}{g'(g^{-1}(0))} < 1 \end{gathered}$$

- By

  (Swartz inequality: bounded holomorphic function on disk is linearly bounded) Let $f : D \to D$ be holomorphic that fixes 0, then $|f(z)|$ is atmost $|z|$, that is,

  $$f \in \mathscr{O}(D, D), f(0) = 0 \implies |f(z)| \leq |z|$$

  Moreover, f is a rotation $f \equiv cz, |c| = 1 \iff$ there exists $z \in D$ such that $|f(z)| = |z|$.

  we conclude

  $$f \circ g^{-1}(w) = cw, |c| = 1$$

- This means

  $$f(z) = cg(z)$$

  but

  $$0 < f'(a) = cg'(a), g'(a) > 0 \implies c = 1$$

  thus $f = g$.

[1]

constructing a biholomorphism into the unit disk

- Assume $0 \notin \Omega$. Then there exists a branch of square root $\sqrt{}$ on Ω. Let the image be $\sqrt{\Omega}$.
  - The square root

    $$\sqrt{} : \Omega \to \sqrt{\Omega}$$

    is bijective with inverse $z \mapsto z^2$.
  - If $w \in \sqrt{\Omega}$ then $-w \notin \sqrt{\Omega}$, as otherwise there are $z_1, z_2 \in \Omega$ such that

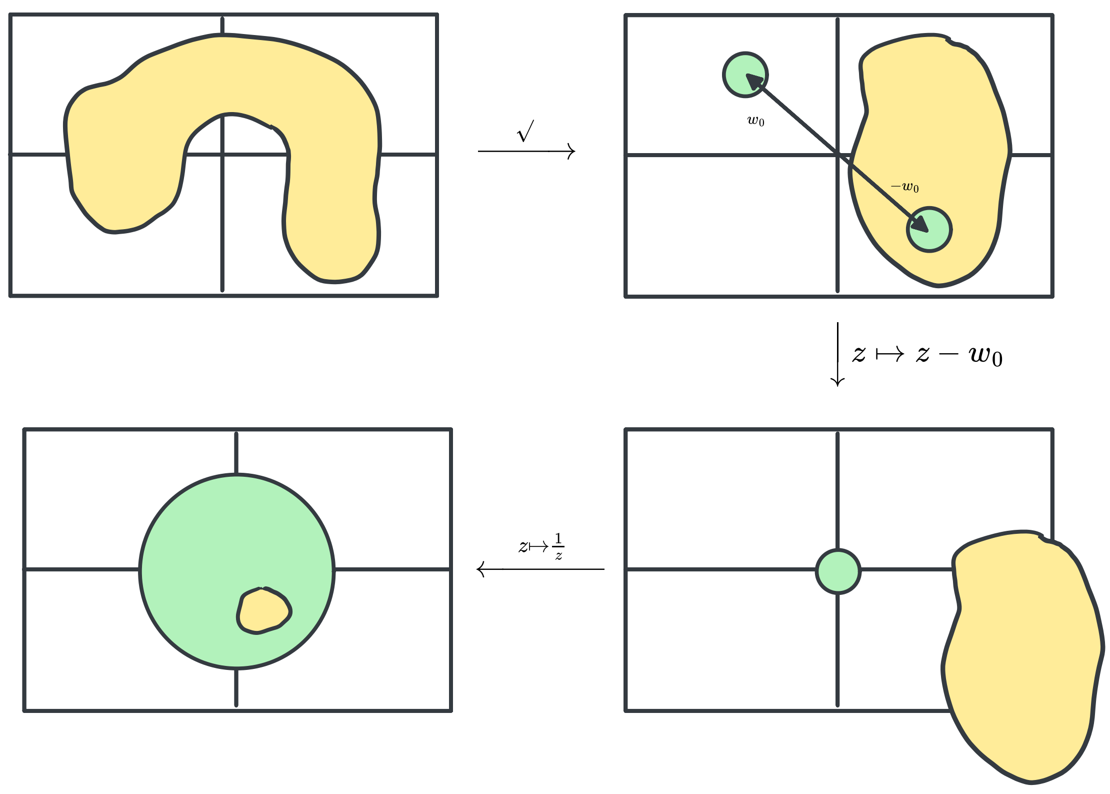
    $$\sqrt{z_1} = w = -\sqrt{z_2} \implies z_1 = z_2 \text{ or } w = -w$$

    which implies $0 \in \Omega$ which is a contradiction.
  - As $\sqrt{\Omega}$ is open so there is a $B_\delta(-w_0) \subseteq \sqrt{\Omega}$ then

    $$\sqrt{\Omega} \cap B_\delta(w_0) = \emptyset$$

- So

$$\left|\sqrt{z} - w_0\right| \geq \delta$$
$$\frac{\delta}{2}\frac{1}{\left|\sqrt{z} - w_0\right|} \leq \frac{1}{2} < 1$$



-
- Thus

$$f : \Omega \to D$$
$$z \mapsto \frac{\delta}{\sqrt{z} - w_0}$$

is injective into the unit disk D.

subjectivity $\iff$ maximality

- *(surjectivity)* For such an Ω, let

$$f : \Omega \to D$$

be an *injective* holomorphic map such that

$$f'(0) = \sup_{g \in \mathcal{F}} |g'(0)|$$

Assume $\alpha \in D \setminus f(\Omega)$.

  - Let $R_\alpha : D \to D$ be an holomorphic automorphism such that $R_\alpha(\alpha) = 0$, then

$$0 \notin R_\alpha(f(\Omega))$$

so $R_\alpha \circ f$ avoids the origin $0 \in D$.

  - As Ω is simply connected, we consider a square root

$$g^2 = R_\alpha \circ f$$

  - Then $h := R_{g(0)} \circ g$ is injective holomorphic map that sends $g(0) \mapsto 0$.
  - Then

$$f = R_\alpha^{-1} \circ (\ )^2 \circ R_{g(0)}^{-1} \circ h$$

  - By ... we have

$$\left|(R_\alpha \circ (\ )^2 \circ R_{g(0)}^{-1})'(0)\right| < 1$$

hence

$$|f'(0)| < |h'(0)|$$

which is a contradiction.

- *(maximality)* Let $f : \Omega \to D$ be a bijective holomorphic map with $f(a) = 0$.
  - For any $h : \Omega \hookrightarrow D$ injective holomorphic with $h(a) = 0$ then

$$g := h \circ f^{-1} : D \to D$$

with $g(0) = 0$

  - By

> **(Swartz inequality: bounded holomorphic function on disk is linearly bounded)** Let $f : D \to D$ be holomorphic that fixes 0, then $|f(z)|$ is atmost $|z|$, that is,
>
> $$f \in \mathscr{O}(D, D), f(0) = 0 \implies |f(z)| \leq |z|$$
>
> Moreover, f is a rotation $f \equiv cz, |c| = 1 \iff$ there exists $z \in D$ such that $|f(z)| = |z|$.

we have

$$|g'(0)| \leq 1$$

  - Now

$$h'(a) = f'(a)g'(0)$$

so

$$|h'(a)| \leq |f'(a)|$$

the family of injective holomorphic maps is pre-compact

- *(existance)* We assume $\Omega \subset D$ and $0 \in \Omega$. Consider the family

$$\mathcal{F} := \{f : \Omega \to D \text{ holomorphic, injective} \mid f(0) = 0, f'(0) > 0\}$$

  - In particular, $f'(0) \in \mathbb{R}$. As $\mathsf{Id}_\Omega \in \mathcal{F}$ it is not empty.

- As

$$|f(z)| < 1 \implies \sup_{z\in D}\sup_{f\in\mathcal{F}} |f(z)| < 1$$

so $\mathcal{F}$ is uniformly bounded on D.

- Consider the sequence

$$f_m \in \mathcal{F}$$

such that

$$f_m'(0) \xrightarrow{m\to\infty} \sup_{f\in\mathcal{F}} |f'(0)|$$

- By

> **(Montel's theorem)** A subset $\mathcal{F} \subseteq \mathscr{O}(U)$ is uniformly bounded
>
> $$\|\mathcal{F}(K)\|_\infty = \sup_{f\in\mathcal{F}} \|f|_K\|_\infty = \sup_{z\in K}\sup_{f\in\mathcal{F}} |f(z)| < \infty$$
>
> on every compact subset $K \subseteq U$ $\iff$ pointwise bounded and equicontinuous
>
> $$\forall a \in U, \|\mathcal{F}(a)\|_\infty < \infty$$
> $$\forall \epsilon > 0 \exists \delta > 0 : \forall z \in \mathcal{F}(B_\delta(x_0)), d(z, f(x_0)) < \epsilon$$
>
> $\iff$ it is **pre-compact** (in the compact-open topology) $\iff$ for every sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U to a holomorphic function in $\mathscr{O}(U)$.

there is a subsequence f_{m_n} which converges

$$f_{m_n} \xrightarrow{n\to\infty} f_\infty \in \mathscr{O}(\Omega)$$

uniformly on compact subsets of Ω, such that

$$f_\infty'(0) = \sup_{f\in\mathcal{F}} |f'(0)|$$

- Then $f_\infty(0) = 0, f_\infty'(0) > 0$, hence the map is not constant.
- As $f_\infty(\Omega) \subset \bar{D}$ by

> **(Open mapping theorem for open subsets in $\mathbb{C}$)** Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

$f(\Omega) \subseteq D$, thus we have a holomorphic

$$f : \Omega \to D$$

by solving a Dirichlet problem

[2]

- Let $\Omega \subset D$ and $0 \in \Omega$. We solve the Dirichlet problem for

$$\log|\zeta| \text{ for } \zeta \in \partial\Omega$$

and obtain a harmonic

$$u : \Omega \to \mathbb{R}$$

such that

$$u(z) \xrightarrow{z\to\zeta\in\partial\Omega} \log|\zeta|$$

  - Since Ω is simply connected there is a harmonic conjugate v to u, making

$$f := z\exp(-(u+iv)) : \Omega \to \mathbb{C}$$

holomorphic on Ω such that

$$|f(z)| \xrightarrow{z\to\zeta\in\partial\Omega} 1$$

  - Hence, $f(\Omega) \subset \bar{D}$ and by open mapping or maximum modulus $f(\Omega) \subset D$. Thus

$$f : \Omega \to D$$

  - Applying the argument principle to ...

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Biholomorphic mapping onto disk

And it has 13 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - Bounding derivative of holomorphic mappings
        - Bounding holomorphic functions linearly
        - Branch points and ramification index of holomorphic functions
        - Disk in holomorphic image of disk
        - Size of holomorphic image of disks
        - Maximum and minimum of holomorphic functions
        - Fibers of holomorphic mappings
        - Non-constant holomorphic functions locally look like w^k and are open mappings
        - Biholomorphic mapping onto disk

- Conformal radius of pointed simply connected open subsets of $\mathbb{C}$
- Holomorphic functions on punchered disk
- Holomorphic function whose singularities are not isolated
- A holomorphic map branched over $\mathbb{Q}$

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1. https://www.math.stonybrook.edu/~bishop/classes/math626.F08/rmt.pdf ↩
2. complex analysis - Solution verification —— Proof of the Riemann mapping theorem using Perron's solution of the Dirichlet problem. - Mathematics Stack Exchange ↩