Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on February 16, 2024 11:13:16 AM,
was last modified on May 17, 2026 7:36:57 PM, and
was exported on September 7, 2026 3:24:50 PM.

Factorization of loops in a topological space with path-connected cover with path-connected intersections

The following are proofs of theorems in Loops and a open cover of a topological space.

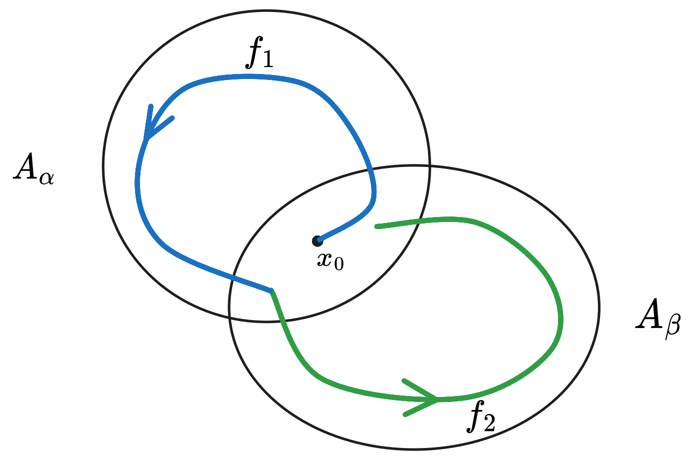
factorization of loops for a path-connected cover with path-connected intersections

- Let

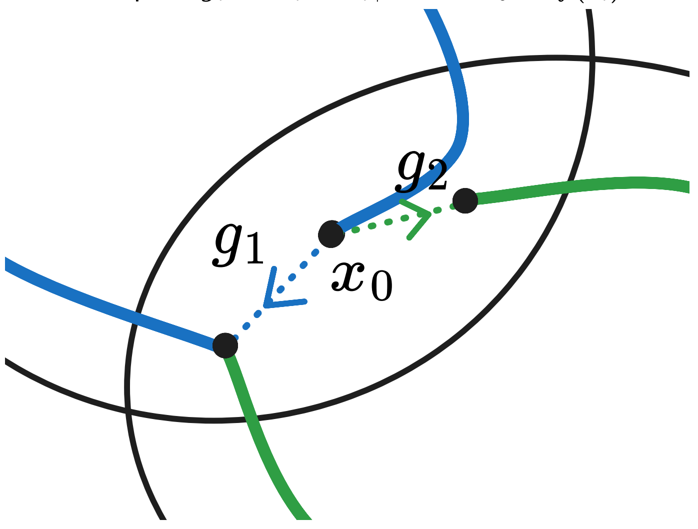
$$X = \bigcup_\alpha A_\alpha$$

Given a loop

$$f : ([0,1], \{0,1\}) \to (X, x_0)$$

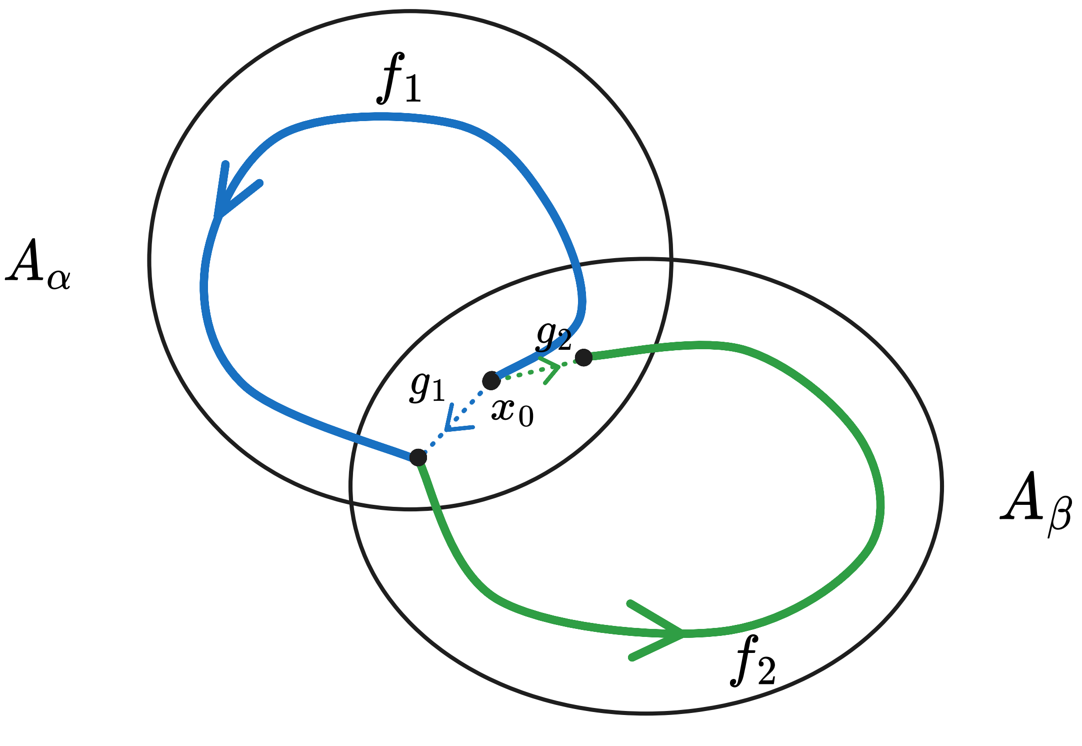
  - There exists a partition $(s_i)_{i=0}^m$ of I such that each $[s_{i-1}, s_i]$ is mapped to a single A_α
    - There exists an interval V_s around each $s \in I$ whose closure is mapped to a single A_α
    - **Compactness** of I implies a finite number of V_s cover I.
  - Denote the A_α containing $f[s_{i-1}, s_i]$ as A_i and f_i be the restriction to that interval.
    - Then $f = f_1 \dots f_m$ where each of f_i is a path in A_i
- Let A_α are *path connected, open with path-connected intersections.*
  - 

  - Choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$.



  - From $f = f_1 \dots f_m$, consider the loop

$$\underbrace{(f_1\bar{g}_1)}_{=:a_1^f}(g_1 f_2 \bar{g}_2)\dots\underbrace{(g_{i-1}f_i\bar{g}_{i+1})}_{=:a_i^f}\dots\underbrace{(g_m f_m)}_{a_m^f}$$



    - The factor a_i^f is a loop lying inside a single A_i such that

$$[f] = [a_1^f]\dots[a_i^f]\dots[a_m^f]$$

    - We note this factorization is *not unique*.
  - Think of $a_i^f : I \to A_i$ as a loop in A_i then

$$[a_i^f] \in \pi_1(A_i)$$

    - Any factorization

$$[b_1]\dots[b_i]\dots[b_m]$$

where each $[b_i] \in \pi_1(A_i)$ determines an element of $\star_\alpha \pi_1(A_\alpha)$

- The inclusions $i_\alpha : A_\alpha \to X$ give the extended induced map

$$\star_\alpha i_\alpha : \star_\alpha A_\alpha \to X$$

then the image of

$$[a_1^f] \dots [a_i^f] \dots [a_m^f] \in \star_\alpha \pi_1(A_\alpha)$$

under this map is

$$[f]$$

- Hence, the map $\star_\alpha i_\alpha : \star_\alpha A_\alpha \to X$ is **surjective** because any $[f] \in \pi_1(X)$ has a factorization like this.

the uniqueness of factorization of loops in a path-connected cover with single and double intersections

- N be the normal subgroup generated by the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$
- Call two factorizations of

$$[f] \in \pi_1(X)$$

in $\star_\alpha \pi_1(A_\alpha)$ *equivalent* if they are related by a sequence of the following movies or their inverses

  - $$[f_i][f_{i+1}] \mapsto [f_i f_{i+1}]$$

  if both $[f_i], [f_{i+1}]$ lie in same $\pi_1(A_\alpha)$
    - this move does not change the element of $\star_\alpha \pi_1(A_\alpha)$ defined by factorization
  - if f_i is a loop in $A_\alpha \cap A_\beta$ then

  $$[f_i] \in \pi_1(A_\alpha) \mapsto [f_i] \in \pi_1(A_{beta})$$

    - this move does not change the image of the element in

    $$\frac{\star_\alpha \pi_1(A_\alpha)}{N}$$

- Any two factorizations of $[f]$ is *equivalent*. Then the map

$$\frac{\star_\alpha \pi_1(A_\alpha)}{N} \to \pi_1(X)$$

is injective and hence $N = \ker \star_\alpha i_\alpha$

  - Let $[a_i^f]$ and $[b_i^f]$ be two factorizations of $[f]$. Then the paths

  $$a_1(a_2(\dots(a_{k-1}a_k)))\dots)$$

  and $b_1 \dots b_l$ are homotopic, say via

  $$F : I \times I \to X$$

  - Parition I into paritions such that F maps each rectangle $[s_{i-1}, s_i] \times [t_{j-1}, tj]$ into a single A_α
  - We order the rectangles from A_i from 1 to mn like
  -

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Loops and a open cover of a topological space
      - Factorization of loops in a topological space with path-connected cover with path-connected intersections

And it has 1 siblings.

- structures based on sets
  - Topological spaces
    - Loops and a open cover of a topological space
      - Factorization of loops in a topological space with path-connected cover with path-connected intersections