ⓘ Info

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Mobius endomorphisms on $\overline{\mathbb{R}^n}^{\infty} \cong S^n$

Definition. Metric on $\overline{\mathbb{R}^n}^{\infty}$

$$d(x,y) := \begin{cases} \dfrac{2\,|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} & x,y \neq \infty \\ \dfrac{2}{\sqrt{1+|x|^2}} & y = \infty \end{cases}$$

which gives the cross-ratio

$$[x,y,u,v] = \frac{|x-u|\,|y-u|}{|x-y|\,|u-v|}$$

that agrees with the cross ratio for the standard metric on $\mathbb{R}^n$

$$x \mapsto \frac{x}{|x|^2}$$

Definition. Mobius endomorphisms

- Let $rS^{n-1}+a$ be the sphere of radius $r>0$ centered at $a \in \mathbb{R}^n$. Then the **inversion through the sphere** is the map

$$\begin{aligned} \mathbb{R}^n \cup \{\infty\} &\to \mathbb{R}^n \cup \{\infty\} \\ x &\mapsto a + \left(\frac{r}{|x-a|}\right)^2 (x-a) \\ &= a + r^2 (x-a)^* \end{aligned}$$

- Let $t + \mathbb{R}\langle a\rangle^{\perp} \cup \{\infty\}$ be the extended plane orthogonal to $a \in \mathbb{R}^n \setminus \{0\}$ at a distance $t \in \mathbb{R}$. The **inversion through the plane** is the map

$$\begin{aligned} \mathbb{R}^n \cup \{\infty\} &\to \mathbb{R}^n \cup \{\infty\} \\ x &\mapsto x - 2(x \cdot a - t)a^* \end{aligned}$$

$$O^+(1,n+1) \cong \mathrm{Moeb}(S^n) \curvearrowright S^n$$

dimension counts

$\dim S^n = n$	$\dim O^+(1,n+1) = (r$	$\alpha(n) \cdot n$	$\mathrm{Moeb}(S^n)_\circ$
2	6	$3 \cdot 2 = 6$	sharply 3-transitive
3	10	$3 \cdot 3 = 9$	
4	15	$3 \cdot 4 = 12$	
5	21	$4 \cdot 5 = 20$	
...			

Current note has 0 direct children and 0 total descendants.

- squishy
  - Mobius endomorphisms

And it has 30 siblings.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
  - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$
  - $\mathrm{Mat}_{\mathbb{C}}(2)$
  - $A_{r},\_\langle z \mapsto \lambda z\rangle \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}}$
  - $\mathbb{C} \setminus \{0\},\_\langle z \mapsto z+1\rangle \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}}$
  - $\mathbb{C} \setminus \{0,1\},\_PSL(2)(\mathbb{Z})[2] \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathrm{U}}}$
  - $\mathbb{R} \ltimes_\theta \mathbb{C}^2$
  - $\boldsymbol{Cl}(\mathbb{R}^n, \langle\ ,\ \rangle_{0,n})$
  - $\mathbb{C}^n$
  - $U(\mathbb{C} \setminus \{0\})$
  - $PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\mathsf{Man}} S^2$
  - $\mathbb{C}P^n$
  - Farey graph, complex and tree
  - GL
  - Heisenberg group
  - Incomplete hyperbolic polygon mod side-pairing
  - $\mathbb{R}\boldsymbol{H}^2,\_\mathbb{R}H^2_{\mathrm{U}},\_D^2$
  - $\mathbb{R}\boldsymbol{H}^n$
  - D_n lattice

- Matrices
- $\mathsf{Mat}_{\mathbb{R}}(2)$
- Mobius endomorphisms
- Elementary group with finite and infinite sided Dirichlet polyhedra
- Polynomials
- $T^*S^1 \ncong_{\mathsf{Sym}} (\mathbb{R}^2 \setminus \{0\}, \mathrm{d}x \wedge \mathrm{d}y)$
- $Z\left(\sum_{1 \leq j \leq n} z_j^2 - 1\right)(\mathbb{C}) \cong_{\mathsf{Sym}} T^*S^{n-1}$
- SL
- SO
- Seifert–Weber dodecahedral 3-manifold
- $U(n, \mathbb{C})$