ⓘ Info

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Harmonic functions composed with conformal maps

Harmonic functions composed with conformal maps is not harmonic in $\dim \geq 3$

Consider the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ with (the metric induced from) the standard metric on $\mathbb{R}^n$. The Laplacian

Definition. We have the Laplace operator

$$\Delta : \mathcal{C}^2 \to \mathcal{C}$$

$$\begin{aligned}
\Delta &:= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\
&= \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} (-) \right) + \frac{1}{r^2} \Delta_{S^{n-1}} \\
&= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}
\end{aligned}$$

Then

$$u(x) = x_1$$

is *harmonic* and the *inversion in* S^{n-1}

$$\begin{aligned}
f : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} &\to \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \\
x &\mapsto \frac{x}{|x|^2}
\end{aligned}$$

is *conformal*. However

$$(u \circ f)(x) = \frac{x_1}{|x|^2}$$

has Laplacian

$$\begin{aligned}
\Delta(u \circ f) &= \frac{\cancelto{0}{\Delta(x_1)}}{|x|^2} + 2 \left\langle \underbrace{\operatorname{grad}(x_1)}_{e_1}, \operatorname{grad}\left(\frac{1}{|x|^2} \right) \right\rangle + x_1 \Delta \left(\frac{1}{|x|^2} \right) \\
&= 2 \frac{-1}{|x|^4} (2x_1) + x_1 \left(\frac{\partial}{\partial r} \frac{-2}{r^3} + \frac{n-1}{r} \left(-\frac{2}{r^3} \right) \right) \\
&= 2 \frac{-1}{|x|^4} (2x_1) + x_1 \left(\frac{6}{r^4} + \frac{n-1}{r} \left(-\frac{2}{r^3} \right) \right) \\
&= \frac{x_1}{|x|^4} (-4 + 6 - 2n + 2) \\
&= \frac{(4-2n)x_1}{|x|^4}
\end{aligned}$$

or

$$\begin{aligned}
\Delta(u \circ f) &= \left(\frac{\partial}{\partial x_1} \right)^2 \frac{x_1}{|x|^2} && + \sum_{i>1} \left(\frac{\partial}{\partial x_i} \right)^2 \frac{x_1}{|x|^2} \\
&= \frac{\partial}{\partial x_1} \frac{1}{|x|^2} - \frac{\partial}{\partial x_1} \frac{2x_1^2}{|x|^4} && - \sum_{i>1} \frac{\partial}{\partial x_i} \frac{x_1}{|x|^4} (2x_i) \\
&= -\frac{2x_1}{|x|^4} - \frac{4x_1}{|x|^4} + \frac{2x_1^2}{|x|^8} (2|x|^2)(2x_1) && - \frac{2x_1}{|x|^4} (n-1) + \frac{x_1}{|x|^8} \sum_{i>1} (4x_i^2)(2|x|^2) \\
&= \frac{x_1}{|x|^4} (-2 - 4 - 2n + 2) && + 8x_1 \frac{\left(\sum_{i \geq 1} x_i^2 \right)^2}{|x|^8} \\
&= \frac{(4-2n)x_1}{|x|^4}
\end{aligned}$$

So it is not harmonic for $n \geq 3$. [1]

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma \backslash^{\mathbb{R}^n}, _\Gamma \backslash^{\mathbb{C}^n}$
    - Harmonic functions composed with conformal maps

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma \backslash^{\mathbb{R}^n}, _\Gamma \backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with C^2 boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions

- Differential forms on $\mathbb{R}^n$
- Fourier-Wigner transform
- Harmonic functions composed with conformal maps
- Hilbert transform
- Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
- Holomorphic subsets of $\mathbb{C}^n$
- Regular hypersurfaces in $\mathbb{R}^{2n}$
- Orientable hypersurfaces in $\mathbb{R}^n$
- Laplace operator on $\mathbb{R}^n$
- $l^\infty(\mathbb{R})$
- Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- Local $\mathbb{R}$-algebras
- $l^p(\mathbb{R})$
- Metric density of subsets of $\mathbb{R}^n$
- Monotone functions on $\mathbb{R}$
- Cauchy integral of periodic functions
- Polygons with integer vertices
- Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$

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1. https://math.stackexchange.com/a/151838/1290493 ↩