Rigidity theorems by Mostow, Sullivan, Tukia, Yue, Ivanov

- Dennis Sullivan, 1979 - The density *at* infinity of a discrete group of hyperbolic motions
- Dennis Sullivan, 1981 - On The Ergodic Theory at Infinity of an Arbitrary Discrete Group of Hyperbolic Motions
- Dennis Sullivan, 1982 - Discrete conformal groups and measurable dynamics
- Dennis Sullivan, 1984 - Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups
- Pekka Tukia, 1985 - Differentiability and rigidity of Möbius groups
- P. Tukia, 1989 - A rigidity theorem for Möbius groups
- Chengbo Yue, 1996 - The Ergodic Theory of Discrete Isometry Groups on Manifolds of Variable Negative Curvature
- Chengbo Yue, 1996 - Mostow rigidity of rank 1 discrete groups with ergodic Bowen–Margulis measure
- Nikolai V. Ivanov, 1996 - Action of Möbius transformations on homeomorphisms: Stability and rigidity
- 📖 Katsuhiko Matsuzaki, Masahiko Taniguchi, 1998 - Hyperbolic Manifolds and Kleinian Groups
- P. Albuquerque, 1999 - Patterson-Sullivan Theory in Higher Rank Symmetric Spaces
- Thomas Roblin, 2003 - Ergodicité et équidistribution en courbure négative
- Marc Bourdon, 2018 - Mostow type rigidity theorems
- M. Gromov, P. Pansu, 1991 - Rigidity of lattices: An introduction
- Edson de Faria, Sebastian van Strien, 2023 - Dennis Sullivan's Work on Dynamics
- Kingshook Biswas, 2015 - On Moebius and conformal maps between boundaries of $\mathrm{CAT}(-1)$ spaces
- Alexandre Minetto, 2025 - Patterson-Sullivan measures and the Prime Geodesic Theorem

Dennis Sullivan, 1979 - The density at infinity of a discrete group of hyperbolic motions

@1979sullivan_density.pdf

> (i) *the radial limit set* $\Lambda_r \subset \Lambda_t$ *has positive* μ*-measure*;
> (ii) *the action of* Γ *on* $(S^d \times S^d)$ *— diagonal is ergodic*;
> (iii) *the geodesic flow on the quotient* $\mathbf{H}^{d+1}/\Gamma$ *is ergodic*;
> (iv) *the Poincaré series diverges at the critical exponent* $\delta(\Gamma) = \delta$;
> (v) *the Markov process on the quotient* $\mathbf{H}^{d+1}/\Gamma$ *is recurrent.*
>
> *Then the first three are equivalent and imply the fourth. If* $\delta > d/2$, *the fourth implies the fifth which implies the first three* (§ 4, § 5, § 7). *So if* $\delta > \frac{d}{2}$ *all five are equivalent.*

On The Ergodic Theory at Infinity of an Arbitrary Discrete Group of Hyperbolic Motions

DS-pub-0053.pdf

Sec 1

Section I. *The variation of area distortion lemma and groups with finite solid angle,*[1] $\sum_{\Gamma} 1/\lambda^2 < \infty$

THEOREM I. *For groups* Γ *of finite solid angle*[1] *in* H^3 *there is on the conservative part of the action of* Γ *on* S^2 *no measurable tangent line field invariant a.e. by* Γ.

Sec 4

THEOREM IV. *The following are equivalent*

i) *The fundamental domain has zero area at* ∞.

ii) *The ratio* $V(r)/H(r) \to 0$ *as* $r \to \infty$.

iii) *The action of* Γ *on the sphere at* ∞ *is conservative.*

iv) *The horospherical limit set of* Γ *has full measure on the sphere.*

Sec 5

THEOREM V. *For a finitely generated discrete subgroup* Γ *of* $PS\ell(2, C)$ *the classes of quasi-conformal deformations consists of a Teichmüller space associated to the uniformized Riemann surface of finite type. For example, if* Γ *has a dense orbit on* S^2, *then* Γ *is quasi-conformally rigid.*

Sec 6

(Sullivan's QC rigidity)

THEOREM VII. *Let* Γ *be a discrete group of conformal transformations of* S^n, $n \geq 2$ *and let* ν *be a measurable conformal structure (on the tangent spaces), which is a.e. invariant by* Γ. *Then* ν *agrees a.e. with the standard conformal structure on the conservative part of action of* Γ *on* S^n.

(Mostow rigidity using Sullivan's QC rigidity) Let M be a complete hyperbolic manifold of dimension ≥ 3 that satisfies any of

THEOREM IV. *The following are equivalent*

i) *The fundamental domain has zero area at* ∞.

ii) *The ratio* $V(r)/H(r) \to 0$ *as* $r \to \infty$.

iii) *The action of* Γ *on the sphere at* ∞ *is conservative.*

iv) *The horospherical limit set of* Γ *has full measure on the sphere.*

then M is Mostow rigid.

Proposition:

COROLLARY. *If* V^3 *is defined by any discrete finitely generated group of isometries of* H^3 *with a dense orbit on* S^2, *then* V^3 *is rigid in the sense of Mostow.*

Dennis Sullivan, 1982 - Discrete conformal groups and measurable dynamics

DS-pub-0055.pdf

(Theorem 4) **The geodesic flow is ergodic relative to a conformal measure μ of dim $\delta \iff \Gamma$ diverges at δ. In this case, the conformal measure satisfying $\gamma^*\mu = |\gamma'|^\delta \mu$ for all $\gamma \in \Gamma$ is unique.**

(Sullivan's measurable rigidity) **Let $\Phi : S^{n-1} \to S^{n-1}$ be a Borel map which is Γ_1, Γ_2-equivariant, and non-singular with respect to conformal densities μ_1, μ_2 of dimension δ that is**

$$\mu_1(A) > 0 \iff \mu_2(\Phi(A)) > 0$$

Then Γ_1 diverges at $\delta \implies \Phi$ is equal a.e. to a conformal conjugate between Γ_1, Γ_2.

Let Γ_1 diverges at δ. By

> **(Theorem 4)** **The geodesic flow is ergodic relative to a conformal measure μ of dim $\delta \iff \Gamma$ diverges at δ. In this case, the conformal measure satisfying $\gamma^*\mu = |\gamma'|^\delta \mu$ for all $\gamma \in \Gamma$ is unique.**

the measure

$$\frac{1}{|x-y|^\delta} \mu_1 \otimes \mu_1$$

is ergodic for $\Gamma_1 \curvearrowright S^{n-1} \times S^{n-1}$.

- Since Φ is non-singular, $\Phi \times \Phi$ is a measurable conjugacy.
- ...
- We conclude

$$(x, y) \mapsto \frac{|\Phi(x) - \Phi(y)|}{|x-y|}$$

  is a.e. a product of function of x and y.
- Thus Φ preserves cross rations for a.e. 4-tuples.
- Thus Φ is equal to an conformal map a.e.

THEOREM 7. *In the recurrent part of the action of a discrete conformal group relative to Lebesgue measure the action on tangent spaces to S^d is irreducible—there is no measurable field of Γ invariant subspaces.*

Applications of Part II. (1) Theorem 7 implies a generalization of the (quasi-conformal → conformal) part of the Mostow rigidity theory.

COROLLARY 2. *Suppose* ϕ: $S^d \to S^d$ *is a quasi-conformal homeomorphism* ($d \geqslant 2$) *conjugating one discrete conformal group* Γ *to another* Γ' ($\gamma \in \Gamma$ *iff* $\phi\gamma\phi^{-1} \in \Gamma'$). *Further suppose the derivative of* ϕ *is a similarity transformation at almost all points of the nonrecurrent part of the action of* Γ. *Then* ϕ *is a conformal transformation*.

SKETCH OF PROOF. If not, the conformal distortion of ϕ on the recurrent part is expressed (in particular) by a Γ invariant measurable family of proper subspaces. Such a family is impossible by Theorem 7. Thus at almost all points the derivative of ϕ is a similarity transformation. Thus by a standard fact in quasi-conformal mappings ϕ is actually conformal. For more details see **[S(ii)**, §7].

citations

Dennis Sullivan, 1984 - Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups

DS-pub-0065.pdf

Pekka Tukia, 1985 - Differentiability and rigidity of Möbius groups

BF01388870.pdf

(Tukia's zooming in)

Theorem A. *Let G be a group of Möbius transformations of $\bar{R}^n$ and let $f: \bar{R}^n \to \bar{R}^n$ be a G-compatible map which is differentiable with a non-vanishing Jacobian at a radial point of G. Then f is a Möbius transformation unless there is a point $z \in \bar{R}^n$ fixed by every $g \in G$. If there is such a point z, then there are Möbius transformations h and h' such that $h(\infty)$ is fixed by every $g \in G$ and that $h' f h | R^n$ is an affine homeomorphism of R^n.*

P. Tukia, 1989 - A rigidity theorem for Möbius groups

@1989tukia_rigidity.pdf

(Tukia's measurable rigidity)

Theorem 3C. *Let $f: A_1 \to A_2$ be a measurable map which is injective and directly measurable outside a null-set with respect to the G_i-measures μ_i of A_i of dimension d_i and which induces a homomorphism $\varphi: G_1 \to G_2$ (in the* a.e. *sense). Suppose that the action of G_1 on $A_1 \times A_1$ is ergodic, that the mass of μ_1 is not concentrated to a point and that at least one of the dimensions $d_i \neq 0$. Then the following alternatives can occur:*

1° *there is a measurable subset $A \subset A_1$ of full measure such that $\mu_2(fA) = 0$, or*

2° *f is* a.e. *the restriction of a Möbius transformation and $d_1 = d_2$. Furthermore, if $S(\mu_1) = S^n$, then φ is a conjugation in the group of all Möbius transformation of S^n.*

Chengbo Yue, 1996 - The Ergodic Theory of Discrete Isometry Groups on Manifolds of Variable Negative Curvature

Yue-ErgodicTheoryDiscrete-1996.pdf

Negatively pinched Hadamard manifolds

- H is a Hadamard manifold with pinched sectional curvature
$$-K_2^2 \leq K \leq -K_1^2, \quad K_2 \geq K_1 > 0.$$
- Γ is a (non-elementary) discrete isometry group acting on H freely and properly discontinuously (which is called Fuchsian).
- ∂H is the ideal boundary of H.
- $M = H/\Gamma$ is a complete Riemannian manifold.
- SH (or SM) is the unit tangent bundle.
- g^t is the geodesic flow on SH (or SM).
- $v(t)$ is the unique geodesic in SH (resp. SM) with the initial velocity $\dot{v}(0) = v \in SH$ (resp. SM).
- $v(\infty)$ (resp. $v(-\infty)$) is the asymptotic class of the geodesic $v(t)$ (resp. $v(-t)$).
- $L(\Gamma)$ is the limit set of Γ on ∂H.
- $L^r(\Gamma)$ is the radical limit set of Γ in $L(\Gamma)$.
- $\Omega = \Omega(\Gamma)$ is the nonwandering set of the geodesic flow on SM

Sec 7

(Sullivan's invariance of crossratios)

(7.1) Nonsingularity, cross ratio, and conformality. Suppose that $M_1 = H_1/\Gamma_1$ and $M_2 = H_2/\Gamma_2$ are two Fuchsian manifolds. Assume that $\varphi : \Gamma_1 \to \Gamma_2$ is a homomorphism and $\Phi : \Lambda_1 = L(\Gamma_1) \to \Lambda_2 = L(\Gamma_2)$ is a Borel map conjugating the action of Γ_1 to that of Γ_2: $\Phi(\gamma\zeta) = \varphi(\gamma)(\Phi(\zeta))$ for all $\gamma \in \Gamma_1$ and $\zeta \in \Lambda_1$. For any Hadamard manifold H we define, for $x \in H$ and $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \partial H$, the function

$$r_x(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \frac{e^{-\beta_x(\zeta_1,\zeta_2)} e^{-\beta_x(\zeta_3,\zeta_4)}}{e^{-\beta_x(\zeta_1,\zeta_3)} e^{-\beta_x(\zeta_2,\zeta_4)}},$$

where β_x is the Busemann cocycle (see (1.2.2)). It is easy to see that r_x does not depend on the choice of the base point x,so we denote it by r. We call $r(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ the cross ratio of $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. This cross ratio coincides with the usual cross ratio in the constant curvature case.

Theorem (7.1.1). *Suppose that $\delta(\Gamma_1) = \delta(\Gamma_2) \triangleq \delta$ and that Φ is nonsingular with respect to the δ-conformal densities σ^1 of Γ_1 and σ^2 of Γ_2. Suppose also that the Poincaré series for Γ_1 diverges at δ. Then Φ preserves the cross ratio almost everywhere.*

Proof. The idea of the following proof is due to Sullivan ([S2]). The map $\Phi \times \Phi$ is a measurable conjugacy between the action Γ_1 on $\Lambda_1 \times \Lambda_1$ and Γ_2 on $\Lambda_2 \times \Lambda_2$. Let U^{σ^i} be the Γ_i-invariant measure on $\Lambda_i \times \Lambda_i$ that we construct in (5.2). The $(\Phi \times \Phi)^*(U^{\sigma^2})$ is a Γ_1-invariant measure on $\Lambda_1 \times \Lambda_1$ absolutely continuous with respect to U^{σ^1} (since $\Phi : (\Lambda_1, \sigma^1) \to (\Lambda_2, \sigma^2)$ is nonsingular). Because the Poincaré series for Γ_1 diverges at δ, the Γ_1 action on $\Lambda_1 \times \Lambda_1$ is ergodic with respect to U^{σ^1}. Thus the Γ_1-invariant function $f = \frac{d(\Phi\times\Phi)^*(U^{\sigma^2})}{dU^{\sigma^1}}$ must be constant almost everywhere on $\Lambda_1 \times \Lambda_1$. Therefore Γ_2 also acts ergodically with respect to U^{σ^2}. If we denote $h(\zeta) = \frac{d\Phi^*(\sigma_y^2)}{d\sigma_x^1}(\zeta)$, where $x \in H_1$, $y \in H_2$, then we have:

$$\begin{aligned} d(\Phi \times \Phi)^*(U^{\sigma^2})(\zeta, \eta) &= e^{-\delta\beta_y(\varphi\zeta,\varphi\eta)} d\Phi^*(\sigma_y^2)(\zeta) d\Phi^*(\sigma_y^2)(\eta) \\ &= e^{-\delta\beta_y(\varphi\zeta,\varphi\eta)} h(\zeta)h(\eta) d\sigma_x^1(\zeta) d\sigma_x^1(\eta). \end{aligned}$$

However $(\Phi \times \Phi)^*(U^{\sigma^2}) = cU^{\sigma^1}$ for some constant c. So we obtain

$$e^{\delta\beta_y(\varphi\zeta,\varphi\eta)} h(\zeta)h(\eta) = ce^{\delta\beta_x(\zeta,\eta)},$$

from which it follows easily that Φ preserves the cross ratio almost everywhere. □

Theorem (7.1.2). *If $\varphi : \Gamma_1 \to \Gamma_2$ is a homomorphism between two Fuchsian groups and $\Phi : L(\Gamma_1) \to L(\Gamma_2)$ a Borel map conjugating the action of Γ_1 to that of Γ_2. Assume that the Poincaré series of Γ_1 diverges at $\delta(\Gamma_1)$ and $\delta(\Gamma_2) = \delta(\Gamma_1)$. Then Φ is conformal almost everywhere with respect to the Busemann metric.*

Chengbo Yue, 1996 - Mostow rigidity of rank 1 discrete groups with ergodic Bowen–Margulis measure

@1996yue_mostow.pdf

Recall that the *radial limit set* $L^r(\Gamma)$ consists of all limit points $\xi \in L(\Gamma)$ such that any geodesic ray joining $x \in H$ and ξ is contained in the ε-neighborhood of the orbit Γx for some $\varepsilon > 0$. One of the main results in [Y1] is the following theorem.

Theorem. *For any non-elementary discrete isometry group Γ, the following properties are equivalent*:

(1) *Γ is of divergent type.*

(2) *$\mu_x(L^r(\Gamma)) > 0$.*

(3) *$L^r(\Gamma)$ has full measure.*

(4) *The diagonal action of Γ on $L(\Gamma) \times L(\Gamma)\backslash$diagonal is ergodic with respect to the measure dU.*

The *convex hull* $H(\Gamma)$ of Γ is defined to be the smallest convex set in $H \cup \partial H$ containing $L(\Gamma)$. The group Γ is called *geometrically finite* if there exist $\varepsilon > 0$ such that the ε-neighborhood of $H(\Gamma)/\Gamma$ in $M \overset{\text{def}}{=} H/\Gamma$ has finite volume and Γ is called *convex cocompact* if $H(\Gamma)/\Gamma$ is compact. It is easy to see that a convex cocompact group is always geometrically finite and a geometrically finite group is convex cocompact if and only if it has no parabolic elements. If Γ is geometrically finite then $L(\Gamma) = L^r(\Gamma) \cup \{\text{countable number of fixed points of parabolic elements in } \Gamma\}$ (see [B]). It is proved in [Y1] that convex cocompact groups are always of divergent type. The main result of [CI] shows that if H is a rank 1 symmetric space, then any geometrically finite group is of divergent type. We can now state our main theorem.

(Yue's measurable rigidity theorem)

Theorem A. *Let Γ_1 and Γ_2 be discrete isometry groups of noncompact symmetric spaces H_1 and H_2 of real rank 1 respectively, such that Γ_1 is*

Zariski dense in $ISO^0(H_1)$ and is of divergent type. Let $\rho : \Gamma_1 \to \Gamma_2$ be a homomorphism. Then there exists a ρ-equivariant map $\varphi : L(\Gamma_1) \to L(\Gamma_2)$ which is measurable with respect to the Patterson–Sullivan measure. Assume that a measurable inverse of φ exists and is nonsingular with respect to the Patterson–Sullivan measures of Γ_1 and Γ_2, then there exists a totally geodesic ρ-equivariant embedding of H_1 into H_2.

If $\Gamma_1, \Gamma_2 \subset SO(n,1)$ and $\delta(\Gamma_1) = \delta(\Gamma_2)$, then the conclusion of Theorem A is due to Sullivan [S3]. Tukia [Tu1] proved that in this case the assumption on the critical exponents is superfluous. However, his difficult argument does not seem to work in the complex or quaternionic case. In this paper, we found a proof which, even for the real case, is much simpler than Tukia's proof.

In fact the condition that Γ_1 is Zariski dense in $ISO^0(H_1)$ is also not a serious restriction. Namely, given any non elementary discrete subgroup Γ_1 there always exists a totally geodesic copy of a noncompact symmetric space H_0 of real rank 1 embedded in H_1 which is invariant under Γ_1 and moreover, Γ_1 is Zariski dense in $ISO^0(H_0)$ (see [CG]).

The existence of the measurable ρ-equivariant map φ follows from the ergodicity and amenability of the Γ_i-action on $L(\Gamma_i)$ and some general theorems of Spatzier, Zimmer and Furstenberg (see [Z], Proposition 4.3.9; [SZ], Theorem 3.1; [F], Proposition 4.2 and also [Ma2], Chapter VI). Theorem A says that such a map is usually singular with respect to the Patterson–Sullivan measures and it is nonsingular if and only if the map is almost everywhere the restriction of a conformal (with respect to the Carnot–Mostow structure) map. In fact, it is true that whenever Γ_1 and Γ_2 are geometrically finite groups and $\rho : \Gamma_1 \to \Gamma_2$ is an isomorphism, there always exists a Borel map $\varphi. L(\Gamma_1) \to L(\Gamma_2)$ which is equivariant with respect to ρ and moreover, φ is continuous and one-to-one outside parabolic fixed point sets (but may be non-continuous at some parabolic fixed points). If ρ is *type-preserving* (i.e., φ carries parabolic elements of Γ_1 bijectively onto parabolic elements of Γ_2), then there exists a unique homeomorphism φ which is equivariant with respect to ρ (see [Tu2], [Tu3]). When both Γ_1 and Γ_2 are convex cocompact and $\rho : \Gamma_1 \to \Gamma_2$ is an isomorphism, the existence of such a homeomorphism φ also follows from the fact that both Γ_1 and Γ_2 are hyperbolic groups in the sense of Gromov. Namely, since $L(\Gamma_1)$, $L(\Gamma_2)$ can be identified with the boundary of Γ_1 and Γ_2 as boundaries of hyperbolic groups, and moreover, since ρ is obviously a quasi-isometry with respect to the word metrics on Γ_1 and Γ_2, it can be extended uniquely to a homeomorphism $\varphi : L(\Gamma_1) \to L(\Gamma_2)$. If both Γ_1 and Γ_2 are lattices, then the existence of the map φ was due Mostow, Prasad and Margulis. In this case, if the dimension of the boundaries of the corresponding rank one symmetric spaces are bigger than one, then φ is quasiconformal with respect to the Carnot–Mostow metric at infinity and hence it is automatically absolutely continuous with respect to the Patterson–Sullivan measures (in this case just the Lebesgue measure). The following result is a corollary of Theorem A and the main theorem in [CI].

Lemma 2.4. *Let Γ_1, Γ_2 be discrete isometry subgroups of $B^n_{\mathbb{F}}(\mathbb{F} = \mathbb{C}$ or $\mathbb{H})$. Assume that Γ_1 is of divergent type, $\rho : \Gamma_1 \to \Gamma_2$ is a homomorphism. Then there is a measurable map $\varphi : L(\Gamma_1) \to L(\Gamma_2)$ equivariant with respect to ρ. If a measurable φ^{-1} exists and is nonsingular with respect to the Patterson–Sullivan measures of Γ_1 and Γ_2 and moreover $\delta(\Gamma_1) = \delta(\Gamma_2)$, then on the Heisenberg model $\mathscr{H}^{n-1}_{\mathbb{F}}$ of $\partial B^n_{\mathbb{F}}$, φ is (up to compositions with transformations in $SU_{\mathbb{F}}(n,1)$) isometric with respect to the distance* (1.8). *In particular, φ can be modified into an isometric bijection: $L(\Gamma_1) \to L(\Gamma_2)$ in the Heisenberg model.*

3. Proof of theorem A (when $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, $\Gamma_1, \Gamma_2 \subset SU_{\mathbb{F}}(n,1)$, $\delta(\Gamma_1) = \delta(\Gamma_2)$)

In this section we prove theorem A assuming in addition that $\Gamma_1, \Gamma_2 \subset SU_{\mathbb{F}}(n,1)$ and $\delta(\Gamma_1) = \delta(\Gamma_2)$. All other assumptions in theorem A are kept. Our starting point is Lemma 2.4. At this stage we would like to point out that given Lemma 2.4, the proof of Theorem A for the real case ($\mathbb{F} = \mathbb{R}, \Gamma_1, \Gamma_2 \subset SO(n,1), \delta(\Gamma_1) = \delta(\Gamma_2)$) is obvious, as every isometric bijection between any two sets in $\mathbb{R}^{n-1}$ must be the restriction of at least one global isometry of $\mathbb{R}^{n-1}$ and, if one set is Zariski dense (or more generally contains at least n points in general position), then the global isometry is also unique. This is the congruence axiom of Euclidean geometry (see [B1]). This axiom is obviously not satisfied by the Heisenberg type geometry we introduced in Sect. 1. The fact that Theorem A is true in the complex or quaternionic case must be due to the special geometric structure of the limit set of a discrete isometry group and the ρ-equivariance of the map φ (as is revealed through our argument). In some sense our limit set is quasi-self similar. Roughly speaking, almost each small piece in the set can be expanded to a standard size and then mapped into the whole limit set by a quasi-isometry. For the proof of our theorem, we also need the existence of a hyperbolic element in Γ_1. Actually, under the above condition (which is satisfied by all discrete subgroups of divergent type), the set of fixed points of all hyperbolic elements in Γ_1 is dense in $L(\Gamma_1)$. Even more is true, the subset of $L(\Gamma_1) \times L(\Gamma_1)$ which consists of the pair of fixed points of hyperbolic elements in Γ_1 is a dense subset of $L(\Gamma_1) \times L(\Gamma_1) \backslash$diagonal (see [E]).

Nikolai V. Ivanov, 1996 - Action of Möbius transformations on homeomorphisms: Stability and rigidity

BF02246768.pdf

Theorem 2.5. *Let Γ be a group of Möbius transformations, and let $h \in \mathcal{N}$ be a point fixed by Γ under the action of* **Möb** *on $\mathcal{N}$. If the homeomorphism h is lipschitz differentiable with a non-zero derivative at a horospherical limit point of Γ, then $h = \mathbf{0}$.*

Sec 3

3.1 Astala–Zinsmeister theorem. A recent theorem of Astala and Zinsmeister ([AsZ]) shows that we cannot replace the lipschitz differentiability by the usual one in Corollary 2.10. Namely, for any fuchsian group Γ such that $\Lambda_C(\Gamma)$ does not have full measure (for such a group Γ the horospherical limit set $\Lambda_H(\Gamma)$ still may have full measure), Astala and Zinsmeister ([AsZ]) constructed an absolutely continuous homeomorphism $f : S^1 \to S^1$ such that $f \circ \Gamma \circ f^{-1}$ is another fuchsian group (the group Γ' of Corollary 2.10), and such that f is not a Möbius transformation. Since f is absolutely continuous, it is differentiable with a non-zero derivative on a set of positive measure. (In fact, homeomorphisms f constructed by Astala and Zinsmeister are more regular than just absolutely continuous: their

Sec 4

Theorem 4.6. *Let Γ be a group of Möbius transformations of the sphere S^n, and let $f : S^n \to S^n$ be a homeomorphism such that $N(f)$ is fixed by Γ under the action of* **Möb** *on $\mathcal{N}$. If f is differentiable with an invertible differential at some conical limit point of Γ, then $N(f) \in \mathbf{0}$. If, moreover, no point of the sphere S^n is fixed by the whole group Γ, then f is a Möbius transformation.*

Theorem 4.9. *If a point $F \in \mathcal{N}$ is fixed by a lattice subgroup of* **Möb** *(i.e. by a discrete subgroup of cofinite volume), then the* **Möb***-orbit of F is relatively compact in $\mathcal{N}$.*

Theorem 4.12. *Suppose that two groups Γ, Γ' of Möbius transformations of S^n are conjugate by a homeomorphism $f : S^n \to S^n$. If Γ is a lattice, then $f \in$* **Möb** *and, hence, Γ and Γ' are conjugate in* **Möb**.

4.14 Horospherical limit points and the Sullivan theorem. Like Theorems 1.3 and 2.2, Theorem 2.5 can be extended to the higher dimensional case. Because there is nothing surprising in the statement or the proof of this generalization, we leave it to the reader.

But we would like to note that a theorem of Sullivan ([Su]) strongly contrasts with the Astala–Zinsmeister theorem discussed in 3.1 and makes this higher dimensional version of Theorem 2.5 less interesting in almost the same way as the Astala–Zinsmeister theorem makes Theorem 2.5 itself more interesting. Namely, the Sullivan ergodic theorem (cf. [Su, Theorem I and Section VII]) implies that if the horospherical limit set of a discrete group Γ of Möbius transformations has full measure and f is a quasiconformal homeomorphism conjugating Γ to some other group of Möbius transformations,

then f has to be a Möbius transformation. Since quasiconformal homeomorphisms are in general only differentiable, but not lipschitz differentiable almost everywhere, in this situation the Sullivan theorem is much stronger than the generalization of Theorem 2.5. (Note that the Astala–Zinsmeister and Tukia examples are quasisymmetric, and quasisymmetric maps are the one-dimensional analogues of the quasiconformal maps in higher dimensions.)

On the other hand, Sullivan's theorem is global, in contrast with the (generalization of) Theorem 2.5 which requires regularity at only one point. No pointwise version of Sullivan's theorem is known.

Katsuhiko Matsuzaki, Masahiko Taniguchi, 1998 - Hyperbolic Manifolds and Kleinian Groups

Katsuhiko Matsuzaki, Masahiko Taniguchi - Hyperbolic Manifolds and Kleinian Groups (1998).pdf

Chapter 1

Definition.

Definition A subgroup Γ of $\mathrm{Isom}^+(\mathbf{H}^3)$ is called a *Kleinian group* if Γ acts on $\mathbf{H}^3$ properly discontinuously. Under the identifications of $\mathrm{Isom}^+(\mathbf{H}^3)$ with $\mathrm{Möb}(B^3)$, $\mathrm{Möb}(H^3)$, Möb and $\mathrm{PSL}_2(\mathbb{C})$, the corresponding subgroups in these groups are also called Kleinian groups.

Chapter 3

3.3.3 *The Mostow rigidity theorem*

As a counterpart of the Marden isomorphism theorem, we prove the following rigidity theorem for geometrically finite Kleinian groups of the first kind. This is a generalization by Margulis, Mostow and Prasad of the original result (Corollary 3.33) due to Mostow (1968).

In this subsection, (quasi) isometries, (quasi) conformal maps and Möbius transformations may be orientation-reversing.

Theorem 3. 32 *Let Γ be a geometrically finite Kleinian group of the first kind, and $\chi : \Gamma \to \Gamma'$ an arbitrary type-preserving isomorphism of Γ onto another Kleinian group Γ'. Then there is an isometric automorphism of $\mathbf{H}^3$ which induces χ. In particular, N_Γ is isometric to $N_{\Gamma'}$.*

Corollary 3. 33 (The Mostow rigidity theorem) *Closed hyperbolic manifolds N_{Γ_1} and N_{Γ_2} are homeomorphic to each other if and only if they are isometric to each other.*

The next theorem after the definition is due to Agard (1983), and is a crucial point in our proof of the rigidity theorem. We introduce a simple proof by Ivanov (1996).

Definition We say that a map $f : D \to \hat{\mathbb{C}}$ of a domain $D \subset \hat{\mathbb{C}}$ is *compatible* with a Kleinian group Γ acting on D if there exists a Möbius transformation γ' for every $\gamma \in \Gamma$ such that $f \circ \gamma = \gamma' \circ f$.

Theorem 3. 34 *Let Γ be a (not necessarily finitely generated) Kleinian group. Assume that $\hat{\mathbb{C}} - \Lambda_c(\Gamma)$ has zero measure. Then every quasiconformal automorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ compatible with Γ is a Möbius transformation.*

Chapter 4

Chapter 5

Theorem 5. 9 *The following conditions are mutually equivalent for a Kleinian group Γ:*

(1) *Γ is conservative;*

(2) *$m(\Lambda_h(\Gamma)) = 1$;*

(3) *For some point a of B^3, $m(P_a^\infty) = 0$;*

(4) *For some point p of N_Γ,*

$$\frac{\mathrm{Vol}\,(\mathcal{U}_r(p))}{\mathrm{Vol}\,(B_r)} \to 0 \quad (r \to \infty)$$

Here B_r is a hyperbolic ball with radius r and $\mathcal{U}_r(p) = \{q \in N_\Gamma \mid d(p,q) < r\}$.

In general, the limit set of a Kleinian group coincides with that of its non-trivial normal subgroup (Lemma 2.22). The following result in (Velling and Matsuzaki 1991) is inspired by this fact.

Theorem 5.12 *Let Γ be a Kleinian group such that the conical limit set $\Lambda_c(\Gamma)$ has full measure on the sphere at infinity. Then every normal subgroup $H \neq \{\mathrm{id}\}$ of Γ is conservative.*

Theorem 5.15 *For a Kleinian group Γ, the following conditions are mutually equivalent:*

(1) $m(\Lambda_c(\Gamma)) = 1$;
(2) Γ *is of divergence type;*
(3) Γ *acts ergodically on* $S^\infty \times S^\infty$.

Furthermore, the following conditions for the hyperbolic manifold N_Γ are also equivalent to those above:

(4) *There exists no Green function on N_Γ with respect to the hyperbolic Laplacian Δ_h;*
(5) *The geodesic flow on the unit-tangent bundle of N_Γ is ergodic.*

[1]

In Section 5.1, we have shown that $\Lambda(\Gamma) = \mathcal{R}(\Gamma)$ for a finitely generated Kleinian group Γ. The aim of this section is to prove the following theorems due to Sullivan (1981).

Theorem 5.17 *Any Beltrami differential for a finitely generated Kleinian group Γ vanishes a.e. on $\Lambda(\Gamma)$.*

Theorem 5.17 is an immediate consequence of the next one, where Γ is not necessarily finitely generated.

Theorem 5.18 *Any Beltrami differential for an arbitrary Kleinian group Γ vanishes a.e. on $\mathcal{R}(\Gamma)$.*

A complete proof is given in Section 5.2.3. Here we only remark on the relationship with the Mostow rigidity theorem. In the proof of Theorem 5.16, we showed that a Γ-invariant measurable line field is always trivial if a Kleinian group Γ acts ergodically on $S^\infty \times S^\infty$. This condition is stronger than that of Γ being conservative:

Proposition 5.19 *If a Kleinian group Γ acts ergodically on S^∞, then Γ is conservative.*

Proof Assume that $m(\mathcal{D}(\Gamma)) > 0$. Then we can take an invariant union of wandering sets $E\,(\subset \mathcal{D}(\Gamma))$ of Γ such that $0 < m(E) < 1$, which implies that the action of Γ is not ergodic. □

Hence we may regard Theorem 5 18 as an extension of Theorem 5.16. Moreover, such an extension of the rigidity theorem is the best possible, stated as in the following Theorem 5.20. We call Theorems 5.17 and 5.18 the *Sullivan rigidity theorems*, and we will call Theorem 5 20 the *Mostow–Sullivan rigidity theorem.*

Theorem 5.20 *A Kleinian group Γ has quasiconformal rigidity if and only if Γ is conservative.*

Proof The sufficiency follows from Theorem 5.18. Conversely, suppose that Γ is not conservative; then there exists an invariant union of wandering sets of Γ with positive measure in $\mathcal{D}(\Gamma)$, and we can make a non-trivial Beltrami differential for Γ on $\mathcal{D}(\Gamma)$. Hence Γ does not have quasiconformal rigidity. □

Finally, we regard Theorem 5.18 as a statement on the rigidity of hyperbolic manifolds.

Proposition 5.21 *Let $N = \mathbf{H}^3/\Gamma$ be a complete hyperbolic manifold for a conservative Kleinian group Γ. If there exists a quasi-isometric homeomorphism f of N onto another $N' = \mathbf{H}^3/\Gamma'$, then f is homotopic to an isometry $N \to N'$. In particular, N is isometrically equivalent to N'.*

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1. 📖 Peter J Nicholls - The ergodic theory of discrete groups ↩

P. Albuquerque, 1999 - Patterson-Sullivan Theory in Higher Rank Symmetric Spaces

PATTERSON-SULLIVAN THEORY IN HIGHER RANK SYMMETRIC SPACES

P. Albuquerque

Abstract

Let $X = G/K$ be a Riemannian symmetric space of noncompact type and Γ a discrete "generic" subgroup of G with critical exponent $\delta(\Gamma)$. Denote by $X_{reg}(\infty)$ the set of regular elements of the geometric boundary $X(\infty)$ of X. We show that the support of all Γ-invariant conformal densities of dimension $\delta(\Gamma)$ on $X_{reg}(\infty)$ (e.g. Patterson-Sullivan densities) lie in a same and single regular G-orbit on $X(\infty)$. This provides information on the large-scale growth of Γ-orbits in X. If in addition we assume Γ to be of divergence type, then there is a unique density of the previous type. Furthermore, we explicitly determine $\delta(\Gamma)$ and this G-orbit for lattices, and show that they are of divergence type.

Quote

Patterson-Sullivan Theory has lead to beautiful rigidity results in the negative curvature setting ([BesCoG], [Y1], [BuMoz]). It has proven very useful in the investigation of discrete subgroups of the isometry group of negatively curved spaces. Our hope is that it will serve similar purposes for higher rank symmetric spaces of noncompact type.

Thomas Roblin, 2003 - Ergodicité et équidistribution en courbure négative

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CHAPTER 1

PRELIMINARIES

This first part sketches a complete introduction to the subject treated, the only required prior knowledge being that of the hyperbolic disk (or Poincaré half-plane). While a good part of the notions and results presented in Sections 1A to 1F are well known in the setting of hyperbolic spaces, or even in that of manifolds of pinched curvature, and sometimes already in that of CAT(-1) spaces, certain generalizations, natural as they may be, do not seem to have been thoroughly examined up to now. We have here endeavored to provide the reader, as far as possible, with the elements necessary so that they may easily reconstruct the theory on which our work is built. We have thought it useful to recall the fundamental results indispensable for what follows, and to insist on their logical sequence. We have also taken care to detail points of proofs that seemed to us insufficiently clarified in the current literature on the subject.

Before beginning Parts 2 to 6, which constitute the main contribution of this work, it is generally not necessary to read the preliminaries in their entirety. They are organized in such a way that it suffices to absorb the common trunk formed by Sections 1A to 1E (one may omit the extended proof of Theorem 1.7 on a first reading) to already be in a position to approach Chapter 2 and then Theorems 3.1, 4.1 and 5.1. Theorem 5.2 requires having become acquainted with Section 1F, whose material is moreover not new in pinched curvature. As for the rest, access to it is governed by Sections 1G for the concepts, and 1H for the technical lemmas.

1A. Geometry

Throughout the present article, we shall consider a proper CAT(-1) metric space X. For everything concerning these spaces, we refer to [Bou] and [G-H]. In order to lighten the notation for the distance between two points x and y in X, we shall dispense with a symbol and simply denote it by (x, y), which, we hope, will not be a source of confusion.

A horosphere in X is a "level set" of a Busemann function β_ξ(x, ·). More precisely, the horosphere based at ξ ∈ ∂X and passing through x ∈ X is the set of points y ∈ X such that β_ξ(x, y) = 0. The function β_ξ in fact defines a (algebraic) "horospherical distance" seen from the base point ξ. The (open) horoball bounded by the horosphere based at ξ ∈ ∂X and passing through x ∈ X is the set of points y ∈ X such that β_ξ(x, y) $>$ 0. One easily sees (by comparison) that closed horoballs are strictly convex. One also checks that, in accordance with the etymology, the sphere of center a ∈ X passing through a fixed point x ∈ X tends, when a $\to$ ξ ∈ ∂X, toward the horosphere based at ξ and containing x.

One calls the Gromov product at x ∈ X of two points a and b in X the quantity

$$\langle a, b\rangle_x = \frac{1}{2}\big[(a, x) + (x, b) - (a, b)\big].$$

For distinct ξ and η in ∂X, one can define ⟨ξ, η⟩_x as the limit of ⟨a, b⟩_x when a $\to$ ξ and b $\to$ η. In fact,

$$\langle \xi, \eta \rangle_x = \frac{1}{2}\left[\beta_\xi(x,p) + \beta_\eta(x,p)\right]$$

for any p ∈ (ξη). One then defines a visual distance d_x on ∂X, seen from the point x ∈ X, by d_x(ξ, η) = e^{−⟨ξ,η⟩_x} for distinct ξ and η in ∂X, and d_x(ξ, ξ) = 0. One can show that d_x is indeed a distance—which is not obvious—inducing the natural topology of ∂X, and that the family of distances {d_x}_{x∈X} is first conformal and second invariant under isometries, that is:

$$d_{x'}(\xi,\eta) = e^{\frac{1}{2}[\beta_\xi(x,x') + \beta_\eta(x,x')]} d_x(\xi,\eta) \tag{1}$$

$$d_{\phi x}(\phi\xi, \phi\eta) = d_x(\xi,\eta). \tag{2}$$

Note also that the d_x-diameter of ∂X equals one (one has moreover ⟨ξ, η⟩_x = 0 if and only if x ∈ (ξη)). It is a consequence, both useful and immediate, of the triangle inequality that ⟨ξ, η⟩_x ≤ r and hence d_x(ξ, η) ≥ e^{−r}, where r is the distance from x to the geodesic (ξη). In this regard, one can see that r = (x, p) for a unique p ∈ (ξη) that is sometimes called the "orthogonal projection" of x onto (ξη).

A final point of notation concerning X: we shall denote by B(x, r) the open ball in X with center x ∈ X and radius r, and by B_x(ξ, r) the open visual ball in ∂X with center ξ and d_x-radius r.

As in hyperbolic space, the isometries of X fall into three disjoint types, aside from the identity id. An elliptic isometry fixes at least one point in X. A parabolic isometry fixes a unique point of X located at infinity; it preserves the horospheres based at this fixed point. Finally, a hyperbolic isometry fixes exactly two points in ∂X; the geodesic joining these two fixed points is called its axis, along which the isometry acts as a translation of displacement $\ell > 0$, called the translation length of the isometry.

The group of isometries of X will be endowed with its natural topology. Throughout the present article, we shall consider a discrete group of isometries Γ. There exist such discrete groups that leave invariant a finite set of points of X; they are called elementary, and, aside from the trivial group, they fall into three disjoint types. A discrete group of isometries of elliptic type possesses at least one global fixed point in $\overline{X}$; it is of finite order and its nontrivial elements are elliptic. A group of parabolic type possesses exactly one global fixed point located at infinity; its nontrivial elements are elliptic (of finite order) or parabolic; such a group preserves the horospheres based at its fixed point. A group of hyperbolic type leaves invariant a unique pair of points of ∂X; such a group contains a distinguished finite-index subgroup, cyclically generated by a hyperbolic isometry; the other nontrivial elements are elliptic. We shall assume throughout that Γ is not elementary. It then contains infinitely many conjugacy classes of primitive hyperbolic isometries, that is, those that are not nontrivial powers of elements of Γ.

The limit set Λ(Γ) of Γ is by definition the set of accumulation points in X of one (of every) orbit of Γ in X. Since Γ is discrete, one has Λ(Γ) ⊂ ∂X. Saying that Γ is non-elementary is equivalent to saying that Λ(Γ) is infinite; then no point of Λ(Γ) is isolated, and Λ(Γ) is the smallest nonempty closed subset of X invariant under Γ. Moreover, Γ acts properly discontinuously on X \ Λ(Γ). The stabilizers of the fixed points of Γ in X, necessarily lying in X ∪ Λ(Γ), are by nature elementary. One assigns to such a fixed point the type (elliptic, parabolic, or hyperbolic) of its stabilizer. The maximal elliptic or parabolic subgroups of Γ are stabilizers of fixed points of Γ.

1B. Conformal densities and invariant densities at infinity

It is the limit set of Γ that is the stage for the dynamics of Γ in X. Thus one has sought to endow this set with measures related to the action of Γ. Since Γ is non-elementary, it is easy to see that there cannot exist nontrivial finite measures on ∂X that are invariant under Γ. Nevertheless, remarkable quasi-invariant measures were invented in [Patt1], and these are the subject of this paragraph. Rather than speaking of measures on the boundary at infinity ∂X, it seems more appropriate here to adopt the concept of density, following [Sull1]. A density (at infinity) will mean an application μ that assigns to each x ∈ X a positive finite measure μ_x on ∂X.

A density μ is said to be conformal of dimension δ ≥ 0 if for all x and x′ in X, the measure $\mu_{x'}$ is absolutely continuous with respect to μ_x, with Radon–Nikodym derivative given by

$$\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta\beta_\xi(x',x)}.$$

A density μ is said to be invariant under Γ if for every γ ∈ Γ and every x ∈ X one has

$$\gamma_*\mu_x = \mu_{\gamma x}.$$

If μ is a conformal density invariant under Γ, then the measure μ_x (x ∈ X) is quasi-invariant under Γ, that is, $\gamma^*\mu_x$ is absolutely continuous with respect to μ_x; but with the concept of density one recovers invariance in the proper sense. Not only will we consider only nonzero conformal densities, but we shall moreover henceforth assume them to be normalized, by imposing that μ_o be a probability (that is, $\|\mu_o\|=1$), where 0 is a point (origin) of X fixed once and for all—this is how we shall be able to speak of uniqueness in certain situations. Furthermore, we choose 0 among the non-elliptic points, that is, those with trivial stabilizer in Γ. Note that the support of a conformal density invariant under Γ necessarily contains Λ(Γ). We shall not, however, require that the support of such a density be exactly Λ(Γ).

The critical exponent δ(Γ) of Γ is a fundamental number defined as the critical exponent of the Dirichlet-type series

$$\sum_{\gamma\in\Gamma} e^{-s(x,\gamma x)} \quad (s \in \mathbb{R}),$$

called the Poincaré series of Γ, which amounts to saying that

$$\delta(\Gamma) = \limsup_{t\to+\infty} \#\{\gamma \in \Gamma \mid (x,\gamma y) \le t\}$$

(the exponent does not depend on the choice of x and y in X); the preceding function of t is generally called the orbital function of Γ—we shall return to it notably in Chapter 4. We shall henceforth assume that δ(Γ) is finite, an assumption that the generality of our geometric framework forces us to state, but which is immediately satisfied notably when X is a pinched curvature manifold (thanks to Rauch's comparison theorem, with a volume argument), or when X admits a cocompact discrete group of isometries.

The foundations of the theory of conformal densities were laid by Patterson ([Patt1], [Patt3]), to whom we owe a celebrated construction, whose details we shall not reproduce, and from which we abstract below the existence statement. The construction essentially consists in extracting a weak limit of a suitably normalized sum of Dirac masses along an orbit of Γ, weighted by the terms of the Poincaré series in s, possibly slightly modified, as s decreases toward δ(Γ). This procedure requires only the continuity of the Busemann function (cf. 1A), and carries over unchanged to CAT(−1) spaces.

Theorem 1.1 (Patterson). — There exists a conformal density of dimension δ(Γ), invariant under Γ and supported on Λ(Γ).

Since we know (see 1A) that there exists no nontrivial measure invariant under Γ at infinity, nor any conformal density of dimension zero invariant under Γ, it follows directly from Theorem 1.1 a fact far from obvious: namely that $\delta(\Gamma)>0$.

From this existence theorem, the theory took remarkable momentum in the work of Sullivan ([Sull1], [Sull2]), whose cornerstone is Lemma 1.3 below, known as the shadow lemma. Along with visual balls, we shall indeed make extensive use of what is called the shadow (at infinity) of a ball B(y,r) in X seen from a point $x \in X$, namely

$$\mathrm{Or}(x,y) = \{\xi \in \partial X \mid]x\xi) \cap B(y,r) \neq \varnothing\}.$$

Let us first record a lemma that follows simply from the triangle inequality, and which we shall mention repeatedly.

Lemma 1.2. — For every $\xi \in \mathrm{Or}(x,y)$, one has

$$(x,y) - 2r < \beta_\xi(x,y) \leq (x,y).$$

Here now is the shadow lemma, originally conceived in constant curvature in [Sull1].

Lemma 1.3 (Sullivan). — Let μ be a conformal density of dimension δ invariant under Γ. Let $x \in X$. Then, as soon as $r>0$ is large enough, there exists $C>0$ such that for every $\gamma \in \Gamma$,

$$\frac{1}{C} e^{-\delta(x,\gamma x)} \leq \mu_x(\mathrm{Or}(x,\gamma x)) \leq C e^{-\delta(x,\gamma x)}.$$

Mostow type rigidity theorems (in Handbook of Group Actions (Vol. IV))

Bourdon. Mostow type rigidity theorems.pdf

Sec 2

Definition 2.13.

1. A metric space Z is *approximately self-similar* if there is a constant $L_0 \geq 1$ such that if $B(z,r) \subset Z$ is an open ball of radius $0 < r \leq \mathrm{diam}(Z)$, then there is an open subset $U \subset Z$ which is L_0-bi-Lipschitz homeomorphic to the rescaled ball $(B(z,r), \frac{1}{r}d)$.

2. A metric space Z is *Ahlfors Q-regular* (for some $Q \in (0,+\infty)$) if there is a measure ν on Z such that for every ball $B \subset Z$ of radius $0 < r \leq \mathrm{diam}(Z)$ one has $\nu(B) \asymp r^Q$.

Note that if Z is Ahlfors Q-regular, then Q is its Hausdorff dimension. Moreover the measure ν and the Q-Hausdorff measure are absolutely continuous with respect to each other, and their Radon-Nikodym derivatives are bounded. Observe also that an approximately self-similar space is Ahlfors Q-regular as soon as its Q-Hausdorff measure is finite and non-zero.

Sec 4

4.2 Sullivan's characterization of Möbius homeomorphisms

The following result is a main ingredient in Sullivan's ergodic approach to Mostow's rigidity and its variants, that will be presented in Section 5. It will also serve repeatedly in the sequel.

Proposition 4.3 ([128])**.** *Suppose (Z_1, d_1) and (Z_2, d_2) are Q-regular metric spaces for some $Q > 0$. Let $\mathcal{H}_1, \mathcal{H}_2$ be the Q-Hausdorff measures of Z_1 and Z_2 respectively. For $i = 1, 2$ let μ_i be the measure*

$$\mu_i(z,w) = \frac{\mathcal{H}_i(z) \times \mathcal{H}_i(w)}{d_i(z,w)^{2Q}}$$

on Z_i^2. Then a homeomorphism $f : Z_1 \to Z_2$ is Möbius if and only if $(f \times f)^* \mu_2 = C\mu_1$ for some constant $C > 0$.

Sec 5 Mostow rigidity

Theorem 5.1. *Let S_1 and S_2 be rank-one symmetric spaces different from $\mathbb{H}^2_{\mathbb{R}}$. For $i = 1, 2$ let Γ_i be a lattice in $\mathrm{Isom}(S_i)$. Then any group isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ is the conjugacy by an isometry from S_1 to S_2.*

Mostow proved the cocompact lattice case [113]. G. Prasad extended the proof to finite covolume lattices [119].

Proof. Suppose first that Γ_1 and Γ_2 are cocompact lattices. Then Γ_i acts geometrically on S_i. By Svarc-Milnor (Proposition 2.3), the isomorphism φ induces a φ-equivariant quasi-isometry from S_1 to S_2. It extends to a φ-equivariant quasi-Möbius homeomorphism $f : \partial S_1 \to \partial S_2$ (see Theorem 2.12). Since S_1 and S_2 are rank-one symmetric spaces different from $\mathbb{H}^2_{\mathbb{R}}$, their boundaries (equipped with the visual metrics d_{CAT}) are Loewner spaces. Thus from Theorem 4.7, the Hausdorff dimensions of ∂S_i are equal and f is absolutely continuous with respect to the Hausdorff measures. Therefore $f \times f$ is absolutely continuous with respect to the measures μ_i defined in Proposition 4.3. Since μ_i is Γ_i-invariant and ergodic (Theorem 2.14), and since $f \times f$ is equivariant, we get that $(f \times f)^* \mu_2 = C\mu_1$ for some constant $C > 0$. Therefore Sullivan's characterization (Proposition 4.3) implies that f is a Möbius homeomorphism. Finally f is the boundary extension of an isometry by Theorem 2.19.

In the non-cocompact case, Prasad [119] proved that the equivariant quasi-isometry from S_1 to S_2 still exists, by using informations about the cusps. The rest of the proof is similar. □

This proof, arising from the Sullivan's ergodic approach [128], is similar in spirit to Mostow's original proof [113]. At the end, Mostow's argument is different, it relies – in a delicate way – on absolute continuity and ergodicity, to show that f preserves the $\mathbb{R}$-circles (i.e. boundaries of curvature -1 totally geodesic planes). Then he deduces from this property that f is the boundary extension of an isometry.

M. Gromov, P. Pansu, 1991 - Rigidity of lattices: An introduction

Rigidity of lattices : An introduction

by M. GROMOV and P. PANSU

We present in these lectures basic facts and ideas in the geometry of discrete subgroups in Lie groups with a special emphasis laid upon the rigidity of lattices in the semi-simple Lie groups. We try to give a broad panorama of the field and explain various approaches to the study of the rigidity. These can be roughly divided into two categories. The first approach developed by G.D. Mostow [1967] and H. Fürstenberg [1967] (see $[Mos]_1$ and $[Für]_1$) uses the geometry and dynamics of the action of discrete subgroups $\Gamma \subset G$ on an appropriate *ideal boundary* of the ambient Lie group G. This geometro-dynamical line of development culminated in the *superrigidity and arithmeticity* theorems for lattices in the simple Lie groups of $\mathbb{R}$-rank ≥ 2, proven by G.A. Margulis in 1974 (see $[Mar]_1$).

The second approach, more analytic in nature, uses elliptic P.D.E. and Bochner type integro-differential inequalities. The first results here obtained by E. Calabi and E. Vesentini [1960] and then by A. Weil [1962] (see [Cal], [Cal-Ves], [Wei]), dealt with local and infinitesimal deformations of $\Gamma \subset G$ and the P.D.E's involved were linear. The Calabi-Vesentini method was delinearized by Y.T. Siu in 1980 who found a Kodaira-Bochner-(Siu) identity for harmonic maps of Kähler manifolds into Riemannian manifolds satisfying certain negative curvature conditions. Using Siu's method, N. Mok [1989] was able to give an alternative proof of Margulis' superrigidity for the (lattices in the isometry groups G of) Hermitian symmetric spaces. More recently, K. Corlette [1990] has found a Bochner formula in the quaternionic case and thus established the superrrigidy for discrete groups of isometries of the quaternionic hyperbolic spaces as well as for the Cayley plane (see $[Cor]_1$). Notice that these cases are not covered by Margulis' theory.

Edson de Faria, Sebastian van Strien, 2023 - Dennis Sullivan's Work on Dynamics

Dennis Sullivan's Work on Dynamics (2306.06290v1).pdf

Lemma 7 *Let $M = \mathbb{H}^n/\Gamma$ be a hyperbolic n-manifold. Then the following assertions are equivalent.*

(i) The ratio $V_M(p,r)/V_{\mathbb{H}^n}(0,r)$ goes to zero as $r \to \infty$.
(ii) The Kleinian group Γ acts conservatively on S_∞.
(iii) The horospherical limit set of Γ has full Lebesgue measure on S_∞.

A fourth assertion equivalent to the above three is that the fundamental domain of Γ in $\mathbb{H}^n$ has zero Lebesgue measure on S_∞. We now have all the necessary elements to state the Sullivan rigidity theorem.

Theorem 8 (Sullivan Rigidity) *Let M be a complete hyperbolic n-manifold, and suppose that M satisfies any of the equivalent conditions of Lemma 7. Then M is Mostow rigid.*

Kingshook Biswas, 2015 - On Moebius and conformal maps between boundaries of $\mathrm{CAT}(-1)$ spaces

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Alexandre Minetto, 2025 - Patterson-Sullivan measures and the Prime Geodesic Theorem

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