Info

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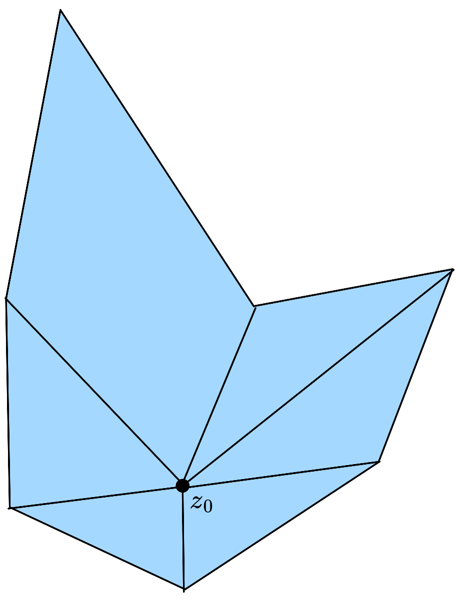
Star-shaped subsets of $\mathbb{R}^n$

Definition. Star-shaped subsets of $\mathbb{R}^n$

A subset $S \subseteq \mathbb{R}^n$ is called **star-shaped** if there exists a point $z_0 \in S$ such that the line segment between z_0 and any point $z \in S$

$$z_0 + t(z_0 - z),\ t \in [0,1]$$

is contained in S.



Then z_0 is called a **center** of S.

examples

- $\mathbb{R}^n$
- convex sets

they are contractible

Star-shaped subsets of $\mathbb{R}^n$ are contractible.

Take a center q of the star-shaped subset, then Id_U is homotopic to the constant map $c_q : x \in U \mapsto q$ by the obvious and natural *straight-line* homotopy:

$$H(x,t) := q + t(x - q)$$

Thus the inclusion $\{q\} \to U$ is a homotopy equivalence.

cohomology of differential forms

For any star-shaped subset U of $\mathbb{R}^n$, its de Rham cohomology are

$$H^k_{dR}(U) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$$

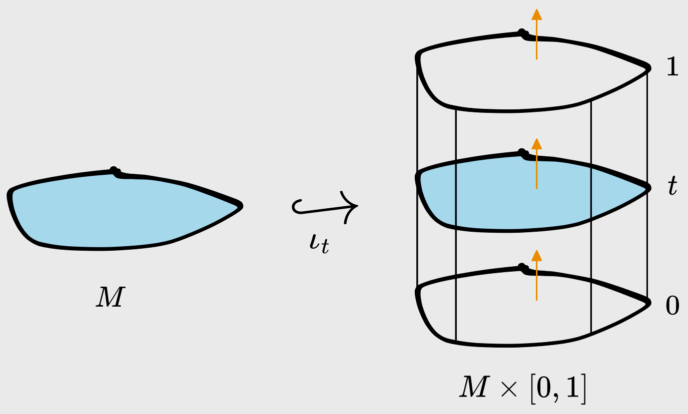
Let $U \subseteq \mathbb{R}^n$ be a star-shaped open set with center at $0 \in \mathbb{R}^n$. Consider differential forms on $U \subseteq \mathbb{R}^{n+1}$ and the homotopy operator

Definition. Homotopy operator from forms on $M \times I$ to forms on M

Let M be a smooth manifold with or without boundary and consider the map

$$\begin{aligned} \iota_t : M &\to M \times [0,1] \\ x &\mapsto (x,t) \end{aligned}$$

for a fixed $t \in [0,1]$.



Let $(-, s) \in M \times [0,1]$ be the coordinate. We define

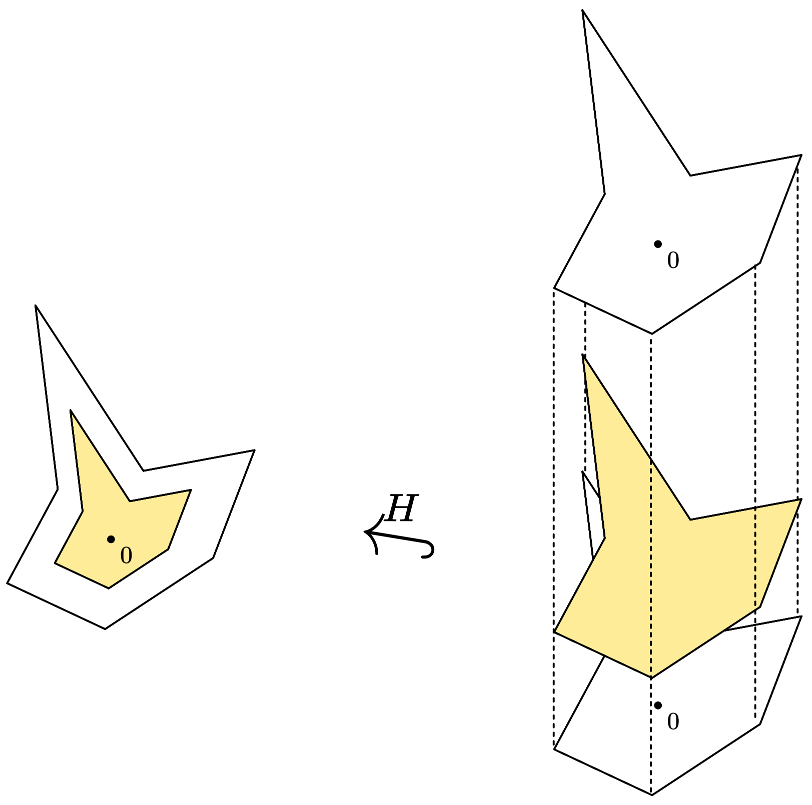
$$\begin{aligned} h : \Omega^k(M \times I) &\to \Omega^{k-1}(M) \\ \omega &\mapsto h\omega \\ (h\omega)_q &:= \int_{t\in[0,1]} \iota_t^*(\iota_{\frac{\partial}{\partial s}}(\omega_{q,t})) \\ (h\omega)_q(-) &= \int_{t\in[0,1]} \omega(\frac{\partial}{\partial s}, d\iota_t(-)) \end{aligned}$$

which satisfies

> Proposition:
>
> $$(\iota_1^* - \iota_0^*)(\omega) = h(\mathrm{d}\omega) + \mathrm{d}(h\omega)$$

Finally with

$$\begin{aligned} H : U \times [0,1] &\to U \\ H(x,t) &= tx \end{aligned}$$



let

$$J := h \circ H^* : \Omega^k(U) \to \Omega^{k-1}(U)$$

then we have

$$\begin{aligned} H^*(\alpha) &= \alpha(\mathrm{d}H(-), \mathrm{d}H(-), \ldots) \\ J(\alpha) = h(H^*(\alpha)) &= \int_{t\in[0,1]} \alpha(\mathrm{d}H(\frac{\partial}{\partial s}), \underbrace{\mathrm{d}H(\mathrm{d}\iota_t(-))}_{t(-)}, \ldots) \\ (J\alpha)_q(w_1, \ldots, w_k) &= \int_{t\in[0,1]} \alpha_{tq}(q, tw_1, tw_2, \ldots, tw_k) \\ (\underbrace{H_1^*}_{\mathsf{Id}} - \underbrace{H_0^*}_{0})(\alpha) &= J\mathrm{d}\alpha + \mathrm{d}J\alpha \end{aligned}$$

Thus we have

$$\alpha = J\mathrm{d}\alpha + \mathrm{d}J\alpha$$

which for $\mathrm{d}\alpha = 0$ gives

$$\alpha = \mathrm{d}(J\alpha)$$

thus we have a anti-derivative for α implying it is exact.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {_\Gamma\backslash^{\mathbb{C}^n}}$
    - Star-shaped subsets of $\mathbb{R}^n$

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {_\Gamma\backslash^{\mathbb{C}^n}}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform
    - Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Holomorphic subsets of $\mathbb{C}^n$
    - Regular hypersurfaces in $\mathbb{R}^{2n}$
    - Orientable hypersurfaces in $\mathbb{R}^n$
    - Laplace operator on $\mathbb{R}^n$
    - $l^\infty(\mathbb{R})$
    - Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - Local $\mathbb{R}$-algebras
    - $l^p(\mathbb{R})$
    - Metric density of subsets of $\mathbb{R}^n$
    - Monotone functions on $\mathbb{R}$
    - Cauchy integral of periodic functions
    - Polygons with integer vertices

- Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$