Info

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Finite cycle decomposition

Definition. Cycles

A k-**cycle** is a sequence

$$(a_1, \dots a_k)$$

of *distinct elements* from $\{1, \dots, n\}, n \geq k$, *equivalent* upto a cyclic reordering

$$(a_1, \dots a_k) \sim (a_l, a_{l+1}, \dots, a_1 \dots a_{l-1}), \forall l < k$$

$$k\text{-cycles}_n := \{k\text{-cycles on } \{1, \dots, n\}\}$$

$$\text{cycles}_n := \bigcup_{k=1}^{n} k\text{-cycles}_n$$

$$n \geq k \implies |k\text{-cycles}_n| = \frac{n!}{(n-k)!(k)}$$

cycle as a permutation

Definition. A cycle

$$(a_1 \dots a_k)$$

defines an element of S_n that maps

$$a_i \mapsto a_{(i+1) \mod n}$$

and fixes everything else.

Let $\sigma \in S_n$ be an m-cycle

> **Definition.** A cycle
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$$\sigma = (a_1 \dots a_m)$$

$$\implies \sigma^i(a_k) = a_{(k+i) \mod n}$$

and hence

$$|\sigma| = m$$

multiplication of cycles

Definition. Multiplication of cycles

We define $\text{cycles}_n \times \text{cycles}_n \to \coprod_{n \geq 2} \text{cycles}_n$

$$(a_1 \dots a_k)(b_1 \dots b_l) = (b_1 \dots)$$

and extend the function to

$$\coprod_{n \geq 2} \text{cycles}_n \times \coprod_{n \geq 2} \text{cycles}_n \to \coprod_{n \geq 2} \text{cycles}_n$$

Disjoint cycles commute.

permutations as multiplication of disjoint cycles

Definition. For a $\sigma \in S_n$, a *cycle of* σ is a cycle

$$(a_1, \dots, a_k)$$

obtained by choosing an a_1 and

$$a_k := \sigma(a_{k-1})$$

The elements of a cycle of σ are just the orbit of the group action

$$\text{Permutation} : S_n \curvearrowright \{1, \dots, n\}$$

Hence, they are disjoint and partition $\{1, \dots, n\}$.

Any

$$\sigma \in S_n$$

can be written as a *multiplication* of **disjoint cycles**

$$\sigma = (a_1 \dots a_n)(b_1 \dots b_n) \dots$$

uniquely upto ordering of the cycles (they commute as they are disjoint) and

$$\sigma_1 = (a_i), \sigma_2 = (b_j) \in S_n \implies \sigma_2 \circ \sigma_1 = (b_j)(a_i)$$

The permutations with only one disjoint cycle

$$\sigma = (a_1 \dots a_n)$$

are the ones coming from

Definition. A cycle

$$(a_1 \dots a_k)$$

defines an element of S_n that maps

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and fixes everything else.

permutations as multiplication of transpositions

Here we break the philosophy of using the orbits as cycles, and factorize the unique disjoint cycle decomposition.

Definition. A **transposition** is a 2-cycle.

Any $\sigma \in S_n$ can be expressed as a product of transpositions.

$$(a_1 \dots a_k) = (a_1 a_k) \dots (a_1 a_4)(a_1 a_3)(a_1 a_2)$$

But they are (generally) not unique and (possibly) not disjoint!

- Let T_i are transpositions and $T_1 \dots T_m = I$ We claim: m is even
  - Assume the claim is true for $m = n - 1$
  - There must be two occurrences of a in the transpositions because it fixed a. Choose both of them from the right, and keep moving them left by using

$$\begin{aligned}(ac)(bc) = (bc)(ab)\\ (ab)(cd) = (cd)(ab)\\ (ab)(bc) = (ac)(ab)\end{aligned}$$

    - If we get

$$(ax)(ax)$$

      on left, we replace it with empty string, and use strong induction hypothesis.
    - If we get

$$(ax)(ay)$$

      ?
    - a cannot appear once on the left

Every *factorization of $\sigma \in S_n$ into transpositions* will **have either an odd number of factors or an even number**.

parity

Definition. We call $\sigma \in S_n$ even/odd if it contains even/odd number of transposition factors.

$$\begin{aligned}\text{Parity} : S_n &\to \{-1, 1\}\\ \sigma &\mapsto \begin{cases} 1 & \text{if even} \\ -1 & \text{if odd}\end{cases}\end{aligned}$$

- Computing the parity
  - A cycle

$$(a_1 \dots a_k) = \underbrace{(a_1 a_k) \dots (a_1 a_4)(a_1 a_3)(a_1 a_2)}_{k-1 \text{ factors}}$$

    - is even if k is odd
    - if odd if k is even
  - $$(a_1 \dots a_l)(b_1 \dots b_k) = \underbrace{(a_1 a_k) \dots (a_1 a_2)(b_1 b_k) \dots (b_1 b_2)}_{k-1+l-1 \text{ factors}}$$
    - (even)(even)=(even)
    - (even)(odd)=(odd)
    - (odd)(even)=(odd)
    - (odd)(odd)=(even)
  - Let $\sigma_1 = C_1 \dots C_r, \sigma_2 = D_1 \dots D_n \in S_n$ be factors of transpositions, then

$$\sigma_1 \sigma_2 = \underbrace{C_1 \dots C_r D_1 \dots D_n}_{r+n \text{ factors}}$$

    - (even)(even)=(even)
    - (even)(odd)=(odd)
    - (odd)(even)=(odd)
    - (odd)(odd)=(even)
  - Hence,

$$\text{Parity}(\sigma_1 \sigma_2) = \text{Parity}(\sigma_1)\text{Parity}(\sigma_2)$$

Therefore,

Definition. We call $\sigma \in S_n$ even/odd if it contains even/odd number of transposition factors.

$$\begin{aligned}\text{Parity} : S_n &\to \{-1, 1\} \\ \sigma &\mapsto \begin{cases} 1 & \text{if even} \\ -1 & \text{if odd} \end{cases}\end{aligned}$$

is a group homomorphism.

The kernel of the homomorphism, the subgroup of all even permutations, is defined as the alternating group.

Current note has 0 direct children and 0 total descendants.

- stretchy
  - Symmetric group
    - S_n
      - Finite cycle decomposition

And it has 2 siblings.

- stretchy
  - Symmetric group
    - S_n
      - Finite cycle decomposition
      - Representations of S_n