Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on February 2, 2026 10:16:59 PM,
was last modified on February 2, 2026 10:17:42 PM, and
was exported on September 7, 2026 3:09:54 PM.

Open subsets of $\mathbb{R}^n$ equipped with the flat metric

Weyl's asymptotic law

Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary and with **Laplacian** $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ let

$$\begin{aligned} \Delta\varphi &= \lambda\varphi \\ \varphi|_{\partial\Omega} &= 0 \end{aligned}$$

be the Helmholtz equation with Dirichlet boundary condition.

- the **spectrum**

$$\begin{aligned} \operatorname{spec}(\Omega) &:= \sigma_{\mathsf{p}}(\Delta : L^2(\Omega) \to L^2(\Omega)) \\ &= \{\lambda \mid \Delta\varphi = \lambda\varphi\} \end{aligned}$$

  is known to be **discrete** with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- the **eigenfunctions** of Δ form an **orthonormal basis** of $L^2(\Omega)$ (when normalized)

(Weyl's asymptotic law, 1911) Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary. Then element λ_k of spectrum of Ω grow like

$$\lambda_k \sim \frac{(2\pi)^2}{(m(D^n)m(\Omega))^{2/n}} k^{2/n} \text{ as } k \to \infty$$

So the spectrum determines the volume!

Definition. Isospectral open sets of $\mathbb{R}^n$

Two open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are called **isospectral** if

$$\operatorname{spec}(\Omega_1) = \operatorname{spec}(\Omega_2)$$

and the multiplicity of each eigenvalue are also equal, that is for each $\lambda \in \operatorname{spec}(\Omega_1) = \operatorname{spec}(\Omega_2)$ we have

$$\dim\ker(\Delta\Big|_{L^2(\Omega_1)} - \lambda\mathrm{Id}) = \dim\ker(\Delta\Big|_{L^2(\Omega_2)} - \lambda\mathrm{Id})$$

> **Proposition:** Isometric open sets are isospectral!

- Monotone functions on $\mathbb{R}$
- Cauchy integral of periodic functions
- Polygons with integer vertices
- Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$