> ⓘ Info
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$\mathscr{O}(\mathbb{C})$

bounded entire functions are constant

- An entire function $f:\mathbb{C}\to\mathbb{C}$ is continuous on every closed disk $\overline{D_R}$ so it **must** attain its maximum on the boundary S^1_R. So for each $z_R \in S^1_R$,

$$z\in\overline{D_R} \implies |f(z)|\le|f(z_R)|$$

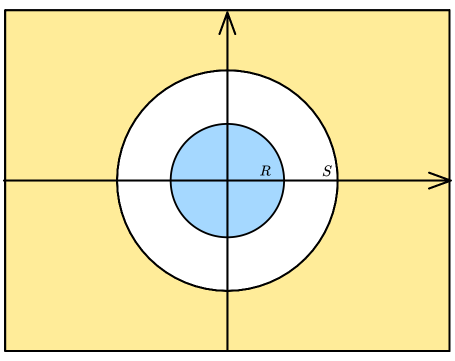
where the equality only holds if f is constant.

- Suppose $f:\mathbb{C}\to\mathbb{C}$ is **entire** and

$$|f(z)| \xrightarrow{|z|\to\infty} 0 \iff \forall\epsilon\exists\delta : |z|>\delta \implies |f(z)|<\epsilon$$

then fix $R>0$ and $z_R\in S^1_R$ and pick $S>R$ such that

$$|z|\ge S \implies |f(z)|\le\frac{|f(z_R)|}{2}$$



If $|f(z_R)|>0$ then inside $D_S\ni z_R$

$$|f(z_R)|\ge\frac{|f(z_R)|}{2}$$

whereas on boundary of $\overline{D_S}$,

$$|f(z)|\le\frac{|f(z_R)|}{2}\le|f(z_R)|$$

which is only possible if f is a constant and identically 0.

- If f is **entire** then

$$\frac{f(z)-f(0)}{z}$$

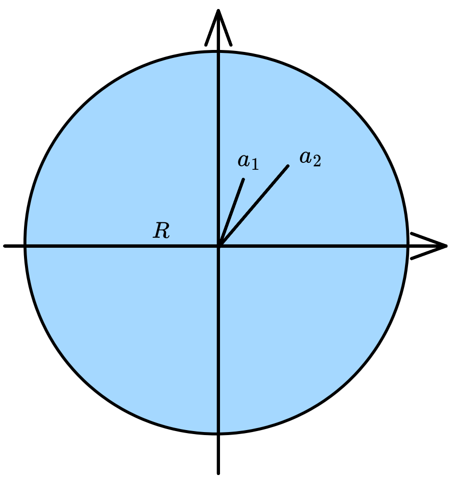
is also entire. If $|f|\le M$ then

$$\left|\frac{f(z)-f(0)}{z}\right|\le\frac{2M}{|z|} \implies \left|\frac{f(z)-f(0)}{z}\right| \xrightarrow{|z|\to\infty} 0$$

which by maximum modulus means $f(z)-f(0)=0$ identically. Thus f is a constant if its bounded and entire.

- Let f is **entire and bounded** by M. For any $a_1,a_2\in\mathbb{C}$ choose any $R>0$ large enough so that

$$|z|=R \implies |z-a_i|\le\frac{R}{2}$$



$$\begin{aligned} f(a_i) &= \frac{1}{2\pi i}\int_{S^1_R}\frac{f(z)\mathrm{d}z}{z-a_i} \\ f(a_1)-f(a_2) &= \frac{a_1-a_2}{2\pi i}\int_{S^1_R}\frac{f(z)\mathrm{d}z}{(z-a_1)(z-a_2)} \\ |f(a_1)-f(a_2)| &\le \frac{|a_1-a_2|}{2\pi}\int_{S^1_R}\underbrace{\left|\frac{f(z)\mathrm{d}z}{(z-a_1)(z-a_2)}\right|}_{\le\frac{M}{R^2}} \end{aligned}$$

$$\le|a_1-a_2|\frac{M}{R}$$

$$\xrightarrow{R\to\infty} 0$$

Then $f(a_1)=f(a_2)$ is true for arbitrary $a_1,a_2\in\mathbb{C}$, hence f must be constant.

- Let

$$f=\sum_{n\ge0}a_nz^n$$

be **entire**.

- Consider

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\int_{S^1_R} \frac{f(z)}{z^{k+1}}\mathrm{d}z$$

for any $R > 0$.

- If f is **bounded** by M then for $k > 0$

$$|a_k| \le \frac{1}{2\pi}\frac{M}{R^k} \xrightarrow{\to\infty} 0$$

- Thus

$$f(z) = a_0$$

a constant.

order at infinity

- By
  - Thus for any function f that is meromorphic at z_0, there is a holomorphic $h : z_0 + D_r \to \mathbb{C}$ and positive integer k called the **order of the pole at** z_0 such that

$$f(z) = \frac{h(z)}{(z-z_0)^k} \text{ on } z_0 + D_r^*$$

Moreover $h(z_0) \neq 0$.

we have for all $w \in D_r^*$

$$f\left(\frac{1}{w}\right) = \frac{h(w)}{w^k} = f(z) = z^n h\left(\frac{1}{z}\right)$$

where h is holomorphic on some D_r, $h(0) \neq 0$ and k is a positive integer called the **order of the pole of** f **at infinity**.

entire functions with a particular maximum modulus

$$\mathscr{O}(\mathbb{C}) \to \mathcal{C}^\infty(0,\infty)$$
$$f \mapsto |f|^2$$

[1]

entire functions that miss two values is constant

Let

$$f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$$

then because

There is a holomorphic covering

$$D \to \mathbb{C} \setminus \{0,1\}$$

we may lift

$$f : \mathbb{C} \to D$$

which is constant because bounded entire functions are constant.

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - $\mathscr{O}(\mathbb{C})$

And it has 19 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - Embedding unit disk D into $\mathbb{C}^3$
      - Factorization of holomorphic functions on $\mathbb{C}$
      - Global holomorphic functions
      - Equivalent descriptions of holomorphic functions
      - Meromorphic functions on the complex plane
      - Holomorphic functions meromorphic at infinity
      - Modular forms
      - Reconstructing meromorphic functions from stalks
      - Extending holomorphic functions by reflections
      - Rotational symmetrization of holomorphic functions
      - Sheaf of holomorphic functions on $\mathbb{C}$
      - $\mathscr{O}(U)$
      - $\mathscr{O}(\mathbb{C})$
      - $\mathscr{O}(D)$
      - $\mathscr{O}^p(H^2_{\mathbb{U}})$
      - $\mathscr{O} \cap L^p$
      - $\mathscr{O}(S^1)$
      - Zeros and singularities of holomorphic functions

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1. citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=6fbe85fa1697efc78b507a1d30a98b5fec32748c ↩