Info

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Image of balls under differentiable maps

Definition. Derivative of a map $U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ at a point

Let

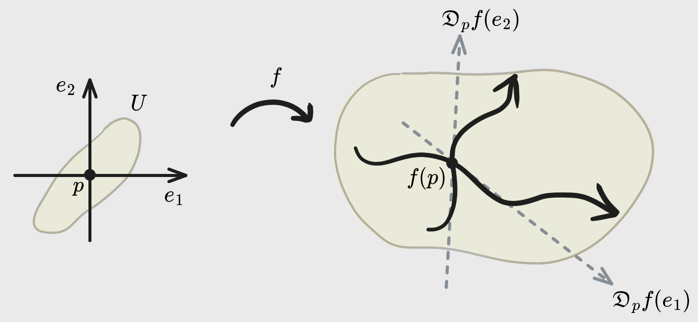
$$f : U \text{ (open)} \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

be a map. Then the **derivative** of f at $p \in U$ is the linear map

$$\mathfrak{D}_p f : T_p\mathbb{R}^n \to T_p\mathbb{R}^m$$

such that

$$\begin{aligned} &\lim_{v\to 0} \frac{f(p+v) - f(p) - \mathfrak{D}_p f(v)}{|v|} = 0 \\ \iff &\lim_{v\to 0} \frac{|f(p+v) - f(p) - \mathfrak{D}_p f(v)|}{|v|} = 0 \end{aligned}$$



$$\begin{aligned} \iff &\lim_{x\to p} \frac{|f(x) - f(p) - \mathfrak{D}_p f(x-p)|}{|x-p|} = 0 \\ \iff &\forall \epsilon > 0 \exists r(\epsilon) > 0 : \\ &\forall x \in B_{r(\epsilon)}(p), |f(x) - f(p) - \mathfrak{D}_p f(x-p)| \le \epsilon\, |x-p| \end{aligned}$$

- This implies for $x \in B_{r(\epsilon)}(p)$ we have

$$\begin{aligned} |f(x) - f(p)| =& |f(x) - f(p) - \mathfrak{D}_p f(x-p) + \mathfrak{D}_p f(x-p)| \\ \le& |f(x) - f(p) - \mathfrak{D}_p f(x-p)| + |\mathfrak{D}_p f(x-p)| \\ \le& \epsilon\, |x-p| + \|\mathfrak{D}_p f\|\, |x-p| \\ \le& (\epsilon + \|\mathfrak{D}_p f\|) r(\epsilon) \end{aligned}$$

  - Therefore, f takes the ball of radius $r(\epsilon)$ about p into the ball of radius $(\epsilon + \|\mathfrak{D}_p f\|) r(\epsilon)$ about $f(p)$

$$B_{r(\epsilon)}(p) \overset{f}{\mapsto} B_{(\epsilon + \|\mathfrak{D}_p f\|) r(\epsilon)}(f(p))$$

for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ with invertible $\mathfrak{D}_p f$

- For all $x \in B_{r(\epsilon)}(p)$ we have

$$\begin{aligned} |f(x) - f(p)| &\ge |\mathfrak{D}_p f(x-p)| - |f(x) - f(p) - \mathfrak{D}_p f(x-p)| \\ &\ge \frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} |x-p| - \epsilon\, |x-p| \\ &\ge \left(\frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} - \epsilon \right) |x-p| \end{aligned}$$

  - Let $\epsilon > 0$ be small enough so that

$$a(\epsilon) := \frac{1}{\|(\mathfrak{D}_p f)^{-1}\|} - \epsilon > 0$$

  - For $x \in \partial B_{r(\epsilon)}(p)$, we have

$$|f(x) - f(p)| \ge \underbrace{a(\epsilon)}_{>0} r(\epsilon)$$

    so

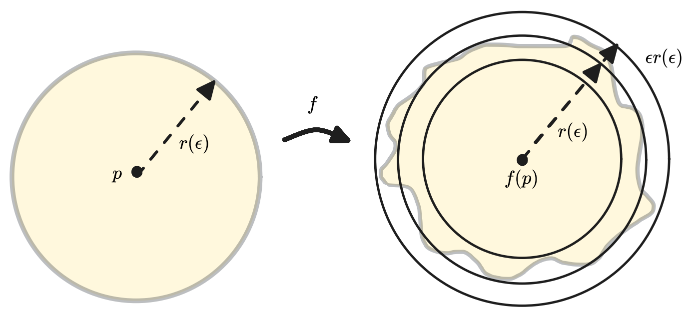
$$\partial B_{r(\epsilon)}(p) \overset{f}{\mapsto} (B_{a(\epsilon) r(\epsilon)} f(p))^{\mathsf{c}}$$

for $\mathfrak{D}_p f = \mathsf{Id}$

- For the case $\mathfrak{D}_p f = \mathsf{Id}$, consider $\epsilon \in (0,1)$, then we have

$$\begin{aligned} B_{r(\epsilon)}(p) &\overset{f}{\mapsto} B_{(\epsilon+1) r(\epsilon)}(f(p)) \\ \partial B_{r(\epsilon)}(p) &\overset{f}{\mapsto} (B_{(1-\epsilon) r(\epsilon)} f(p))^{\mathsf{c}} \end{aligned}$$

  - Therefore, $f(x)$ is strictly different from $f(p)$ outside the ball of radius $(1-\epsilon) r(\epsilon)$ about $f(p)$.



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Intuition

The ball $B_{(1-\epsilon)r(\epsilon)}(f(p))$ seems to be inside the image $f(B_{r(\epsilon)}(p))$ by the above picture.

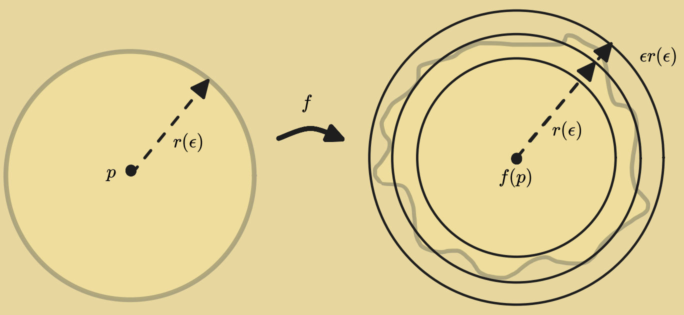
And it is.

Let $p \in U$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a map such that $f(p) = p, \mathfrak{D}_p f = \mathrm{Id}$, that is,

$$\forall \epsilon \exists r(\epsilon) : \forall x \in B_{r(\epsilon)}(p), \left| f(x) - \cancel{f(p)} - x - \cancel{p} \right| \leq \epsilon \left| x - p \right|$$

Then for each $r \in (0, r(\epsilon)]$

$$B_{(1-\epsilon)r}(f(p)) \subseteq f(B_r(p))$$



For $r \in (0, r(\epsilon)]$ and $x \in B_r(p)$ and $y \in B_{(1-\epsilon)r}(p)$ we have

$$\begin{aligned}
|x - f(x) + y - p| &\leq |x - f(x)| + |y - p| \\
&\leq \epsilon \left| x - p \right| + |y - p| \\
&\leq \epsilon r + (1 - \epsilon) r \\
&= r
\end{aligned}$$

- Then

$$\begin{aligned}
\mathfrak{P}_y f : \overline{B_r(p)} &\to B_r(p) \\
x &\mapsto x - f(x) + y
\end{aligned}$$

- By

  (Brouwer's fixed point theorem) Let $\varphi : \overline{D^n} \to \overline{D^n}$ be a continuous map for $n \geq 2$, then it has a fixed point.

  we conclude $\mathfrak{P}_y f$ has a fixed point in $\overline{B_r(p)}$.

- Then

$$\exists x : x = x - f(x) + y \implies f(x) = y$$

- Thus,

$$B_{(1-\epsilon)r}(p) \subseteq f(B_r(p))$$

for conformal $\mathfrak{D}_p f$

for $\mathbb{C}$-linear $\mathfrak{D}_p f$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash^{\mathbb{R}^n}, \Gamma \backslash^{\mathbb{C}^n}$
    - mapping
      - Image of balls under differentiable maps

And it has 5 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash^{\mathbb{R}^n}, \Gamma \backslash^{\mathbb{C}^n}$
    - mapping
      - Image of balls under differentiable maps
      - Inverse of differentiable maps
      - Mean value property of functions
      - Operator norm of linear maps between $\mathbb{R}^n$
      - Sequence of functions on $\mathbb{R}^d$