$\mathbb{Q}_{\text{constr}}$

$$\mathbb{Q} \leq \mathbb{Q}_{\text{Hilbert}} \leq \mathbb{Q}_{\text{constr}} \leq \mathbb{R} \cap \bar{\mathbb{Q}} \quad \begin{matrix} \leq \mathbb{R} \leq \mathbb{C} \\ \leq \bar{\mathbb{Q}} \end{matrix}$$

Proposition 16.3
Let Ω be the set of all real numbers that can be expressed starting from the rational numbers and using a finite number of operations $+, -, \cdot, \div$, and $c \mapsto \sqrt{1+c^2}$. (Note that for any $c \in \mathbb{R}$, $1 + c^2 > 0$, so $\sqrt{1+c^2} \in \mathbb{R}$.) Then Ω is an ordered Pythagorean field.

Proof To show that Ω is a field, let $a, b \in \Omega$. Then each of a, b can be expressed in a finite number of steps using rational numbers and operations $+, -, \cdot, \div$, $c \mapsto \sqrt{1+c^2}$. Hence the same is true of $a \pm b$, $a \cdot b$, and a/b, provided that $b \neq 0$. If c is any element of Ω, then c can be expressed in a finite number of such steps, so $\sqrt{1+c^2}$ can also, and so $\sqrt{1+c^2} \in \Omega$. Hence Ω is Pythagorean. Ω is an ordered field, because it is a subfield of $\mathbb{R}$, so we can take P to be those elements of Ω that are positive as real numbers.

Remark 16.3.1
Clearly, Ω is the smallest Pythagorean ordered field. We call it *Hilbert's field*, since he studied it in his *Foundations of Geometry* (1971). It is also the smallest field over which all of Hilbert's axioms of betweenness and congruence will hold (17.3).

Proposition 16.4
Let K be the set of all real numbers that can be obtained from the rational numbers by a finite number of operations $+, -, \cdot, \div$, and $a > 0 \mapsto \sqrt{a}$. Then K is a Euclidean ordered field.

Proof Similar to the proof to (16.3). Note that we may take square roots only of positive elements. Since K is also a subfield of $\mathbb{R}$, we get the ordering on K from $\mathbb{R}$ as above.

Current note has 0 direct children and 0 total descendants.

- stretchy
  - $\mathbb{Q}_{\text{constr}}$

And it has 62 siblings.

- stretchy
  - Trefoil 3_1 knot
  - The only compact connected Lie group that can act faithfully on a torus is itself
  - Hecke algebras
  - B^n
  - $\mathrm{BS}(1,2)$
  - $(\mathbb{C}, +, \times)$
  - $D^* := D \setminus \{0\}$
  - Cantor set
  - Configuration space of N rigid points in $\mathbb{R}^3$
  - Free groups
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
  - H_2^3
  - $[a, b]$
  - Figure 8 knot
  - Lamplighter group on $\mathbb{Z}_2$
  - $\beta\mathbb{N}$
  - $\#_2\mathbb{R}P^2$
  - $\prod_{i \in \mathbb{N}} \{1, \ldots, n_i\}$
  - $\mathbb{R}P^n$
  - $(\mathbb{Q}, +, \times, <)$
  - $\mathbb{Q}_{\bar{p}}$
  - $\overline{\mathbb{Q}}$
  - $\mathbb{Q}_{\text{constr}}$
  - $\mathbb{Q}(\xi_n)$
  - $\mathcal{O}_{\overline{\mathbb{Q}}}$
  - $(\mathbb{R}, +, \times, <, \sup)$
  - Homeomorphism of $\mathbb{R}^2$ that sends $\mathbb{N}$ to a countable closed discrete subset
  - $\mathbb{R}^2 \times S^1$
  - $\mathbb{R}^n$
  - $\mathbb{R} \times S^1$
  - $\mathbb{R} \times S^2$
  - S^1
  - $S^1 \wedge S^1$

- $S^1 \times S^n$
- S^2 and $\mathbb{C}\boldsymbol{P}^1$
- Mapping torus of the antipodal map of S^2
- S^3
- S^n
- I-bundle
- Symmetric group
- T
- T^2
- T^2_g
- $T^2_{0,0,3}$
- $\mathbb{R}^2 \setminus \{0\}$
- Three $T^2_{1,0,1}$ glued at their boundaries
- $T^2_{1,1,0}$
- T^2_2
- $T^2_g \times [0, 1]$
- $\#_{\mathbb{N}} T^2$
- Thompson's group
- T^n
- 3-valent tree
- 4-valent tree
- $[0, 1]^{\mathbb{N}}$ with product topology
- Whitehead manifolds
- $\mathbb{Z}$
- $\mathbb{Z}[\sqrt{m}]$
- $\mathbb{Z}[X]$
- $\mathbb{Z}^{\mathbb{N}}$
- $\mathbb{Z}[i]$
- $\mathbb{Z}[i\sqrt{5}]$