ⓘ Info

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Path groupoid of a topological space

Definition. Category of paths in a topological space X

Let $X \in \mathsf{Top}$. Then define the category of paths upto reparameterization $\mathsf{Path}(X)$ with

- *objects* are elements of X

$$\mathrm{Obj}(\mathsf{Path}(X)) = X$$

- *morphisms* are paths upto reparameterization

$$\mathrm{Hom}\mathsf{Path}(x_0, x_1) := \frac{\{\text{paths } p : [0,1] \to X | p(0) = x_0, p(1) = x_1\}}{p \sim \alpha \circ p \text{ for } \alpha : [0,1] \to [0,1], \alpha(0) = 0, \alpha(1) = 1}$$

- *morphism composition* is concatenation of paths
- *identity morphism*

Fundamental path groupoid

Definition. Fundamental path groupoid of a topological space

Let $X \in \mathsf{Top}$. Then define the **path groupoid of** X, $\Pi_1(X)$ with

- *objects* are the elements of X

$$\mathrm{Obj}(\Pi_1(X)) := X$$

- *morphisms* are homotopy classes of paths relative to endpoints

$$\mathrm{Hom}\Pi_1(x_0, x_1) := \frac{\{\text{ paths } p : [0,1] \to X : p(0) = x_1, p(1) = x_1\}}{\text{homotopy rel } \{0,1\}}$$

- *morphism composition* is the concatenation of paths upto homotopy equivalence

$$[p_1] \circ [p_2] = [p_1 \cdot p_2]$$

Proposition:

Definition. Fundamental path groupoid of a topological space

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- *morphism composition* is the concatenation of paths upto homotopy equivalence

$$[p_1] \circ [p_2] = [p_1 \cdot p_2]$$

is a groupoid with

$$\mathsf{Id}_x = [c_x]$$

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Topological spaces
    - Paths in a topological space
      - Path groupoid of a topological space

And it has 3 siblings.

- structures based on sets
  - Topological spaces
    - Paths in a topological space
      - Concatenation of paths in a topological space
      - Fundamental group of a topological space
      - Path groupoid of a topological space