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# Quasiconformal maps on $S^n \cong \overline{\mathbb{R}^n}^\infty$

**Definition.** Definition

$$L_{r,x}(f) := \sup \{|f(x) - f(y)| \, |x-y| \leq r\}$$

This means

$$f(\overline{B(r,x)}) \subseteq \overline{B(L_{r,x}(f), f(x))}$$

and

$$B(l_{r,x}(f), f(x)) \subseteq f(\overline{B(r,x)})$$

Image of balls under differentiable maps

The ratio

$$\frac{L_{r,x}(f)}{l_{r,x}(f)}$$

denotes how much the image $f(\overline{B(r,x)})$ gets stretched. As $r \to 0$ the limit denotes how stretched the image of infenitesimal closed balls is. This quantity is called the *linear dilation* of $f$ at $x$.

**Definition.** Linear dilation

Consider a homeomorphism $f : U_1 \subseteq \mathbb{R}^n \to U_2 \subseteq \mathbb{R}^n$ between two open subsets $U_1, U_2$. The **linear dilation** of $f$ at $x \in U_1$ is

$$\begin{aligned} H_x(f) &:= \limsup_{r\to 0} \sup \left\{ \frac{|f(x)-f(y)|}{|f(x)-f(z)|} \middle| |y-z| = r \right\} \\ &= \limsup_{r\to 0} \frac{\sup \{|f(x)-f(y)| | |x-y| \leq r\}}{\inf \{|f(x)-f(y)| | |y-x| \geq r\}} \end{aligned}$$

Conformal maps have linear dilation equal to 1 everywhere. Quasiconformal maps are homeomorphism which have a bounded linear dilation.

**Definition.** Quasiconformal maps between open subsets of $\mathbb{R}^n$

The homeomorphism $f$ is **quasiconformal** if it has a bounded linear dilation $\sup_{x\in U_1} H_x(f) < \infty$.

$$\|\mathfrak{D}_x f\| \leq M$$

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> $$\sup_{y,z\in S_r(x)} \frac{|f(x)-f(y)|}{|f(x)-f(z)|} = \frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}}$$

$$\sup_{y\in S_r(x)} \leq \sup_{y\in B_r(x)}$$

and

$$\inf_{y\notin B_r(x)} \leq \inf_{y\in S_r(x)}$$

$$|f(x)-f(y)| \leq M(r_k)$$

$$\frac{1}{|f(x)-f(z_k)|} \leq \frac{1}{m(s_k)}$$

$$\frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}} \leq \frac{\sup_{y\in B_r(x)}}{\inf_{z\notin B_r(x)}} \leq \limsup_{r\to 0} \frac{\sup_{y\in S_r(x)}}{\inf_{z\in S_r(x)}}$$

## quasisymmetric mappings

## 3 Quasisymmetric mappings

By Theorem 2.1 we know that quasiconformality implies the uniform local boundedness of $H_f(x,r)$. We introduce the equivalent concept of quasisymmetry that turns out to be very useful.

Let $X$ and $Y$ be metric spaces and let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. A homeomorphism $f : X \to Y$ is $\eta$-quasisymmetric ($\eta$-qs), if

$$\frac{|f(a)-f(x)|}{|f(b)-f(x)|} \leq \eta\left(\frac{|a-x|}{|b-x|}\right)$$

for all $a \neq x \neq b$.

**3.1 Remark.** If $f$ is $\eta$-quasisymmetric, then

$$H_f(x,r) = \frac{L_f(x,r)}{l_f(x,r)} \leq \eta(1).$$

So, quasisymmetric mappings are quasiconformal.

We next prove that quasiconformal mappings are locally quasisymmetric.

**3.2 Theorem.** Let $f : B(x_0, 3r_0) \to \Omega' \subset \mathbb{R}^n$ be a homeomorphism such that $H_f(x,r) \leq H$ for all $x \in B(x_0,r_0)$ and $0 < r < 2r_0$. Then $f_{|B(x_0,r_0)}$ is $\eta$-quasisymmetric, where $\eta$ depends only on $n$ and $H$.

## regularity of quasiconformal mappings

**4.1 Proposition.** Let $f : \Omega \to \Omega'$ be a homeomorphism. Then

$$\mu_f'(x) = \lim_{r\to 0} \frac{|f(\overline{B}(x,r))|}{|B(x,r)|}$$

exists almost everywhere in $\Omega$, belongs to $L^1_{\text{loc}}(\Omega)$ and

$$\int_E \mu_f'(x)\, dx \leq |f(E)|$$

for each Borel set $E \subset \Omega$, with equality whenever $|A| = 0$ implies $|f(A)| = 0$.

Definition.

$$L_x(f) := \limsup_{r\to 0} \frac{\sup\{|f(x)-f(y)|\}}{r}$$

Let $f : \Omega_1 \subseteq \mathbb{R}^n \to \Omega_2 \subseteq \mathbb{R}^n$ be a homeomorphism. The function

$$L_x(f) := \limsup_{r\to 0} \frac{L_{r,x}(f)}{r}$$

is Borel measurable and

$$\mu_f'(x) \leq L_x(f)^n \leq H_f(x)^n \mu_f'(x)$$

for almost every $x \in \Omega_1$. In particular, if $f$ is quasiconformal then $L_f \in L^n_{\text{loc}}(\Omega_1)$.

[1]

Consider the closed subsets

$$A_n := \left\{ x \in E \middle| \forall 0 < |h| < \frac{d(E,\partial\Omega)}{n}, \frac{|f(x+h)-f(x)|}{|h|} \leq t - \frac{1}{n} \right\}$$

then for compact $E$

$$E \cap \{L_f < t\} = \bigcup_{n\geq 1} A_n$$

- Let $x \in \Omega$ and $r \in (0, d(x,\partial\Omega))$. Then

$$\begin{aligned}
f(\overline{B(r,x)}) &\subseteq \overline{B(f(x), L_{r,x}(f))} \\
\implies m(f(\overline{B(r,x)})) &\leq m(B(0,1)) L_{r,x}(f)^n \\
\implies \frac{m(f(\overline{B(r,x)}))}{m(B(r,x))} &\leq \frac{L_{r,x}(f)^n}{r^n} \\
&\leq \left(\frac{L_f}{l_f}\right)^n \left(\frac{l_{r,x}(f)}{r}\right)^n \\
&\leq \left(\frac{L_f}{l_f}\right)^n \frac{m(f(\overline{B(r,x)}))}{m(B(r,x))}
\end{aligned}$$

**4.10 Lemma. (Reverse Hölder Inequality)** Let $f$ be $\eta$-quasisymmetric on $2B \subset \mathbb{R}^n$. Then

$$\left(\fint_B L_f^n\right)^{1/n} \leq C(n,\eta) \fint_B L_f.$$

**4.12 Corollary.** Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$. Then $|f(E)| = 0$, if and only if $|E| = 0$. In particular,

$$|f(E)| = \int_E \mu_f'\, dx,$$

for Borel (and all Lebesgue measurable) sets $E$, and $f$ maps Lebesgue measurable sets to Lebesgue measurable sets. Moreover, $\mu_f'(x) > 0$ almost everywhere.

- Let $m(E) = 0$.
  - Given $\epsilon > 0$ consider an open set $V$ with $E \subset V \subset U$ and $m(V) < \epsilon$.

- For each $x \in E$ we pick

$$\overline{B_{r(x)}(x)}$$

such that $\overline{B_{15r(x)}(x)} \subset V$.

$$f(E) \subset f\left(\bigcup_j 5\overline{B_j}\right)$$

- $$\left|f\left(\bigcup_i 5\overline{B_j}\right)\right| \le \sum_j \left|f\left(5\overline{B_j}\right)\right| \le \eta(5)c(n)\sum_j L_f(x_j,r_j)^n$$

- ...

$$m\left(f\left(\bigcup_j 5\overline{B_j}\right)\right) \le A(n,\eta)\sum_j \int_{B_j}(Lf)^n \le A(n,\eta)\int_{\bigcup_j B_j}(Lf)^n$$

- Letting $\epsilon \to 0$ we conclude $m(f(E)) = 0$.

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## differentiable almost everywhere

**4.28 Corollary.** Let $f : \Omega \to \Omega'$ be quasiconformal, where $\Omega, \Omega' \subset \mathbb{R}^n$, $n \ge 2$, are domains. Then $f$ belongs to $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$ for some $p > n$ and, for almost every $x \in \Omega$, $f$ is differentiable at $x$ with $J_f(x) \ne 0$ and satisfies

$$|Df(x)|^n \le H_f(x)^{n-1}|J_f(x)|.$$

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Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, _\Gamma\backslash^{\mathbb{C}^n}$
    - quasiconformal
      - Quasiconformal maps on $S^n$

And it has 2 siblings.

- stem
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    - quasiconformal
      - Quasiconformal maps on $\mathbb{C}$
      - Quasiconformal maps on $S^n$

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1. quasifinal, p.30 ↩