

Info

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on May 26, 2026 4:27:48 PM, was last modified on May 27, 2026 8:12:57 PM, and was exported on September 7, 2026 3:09:55 PM.

Calderon-Zygmund decomposition

dyadic decomposition of cubes and $\mathbb{R}^d$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $R \subseteq \mathbb{R}^d$ be a cube. Then for every $t \in \mathbb{R}$

$$\frac{1}{m(R)} \int_R |f| \geq t$$

there are countably many disjoint dyadic subcubes $\{Q_j\}$ of R such that

$$2^n t \geq \frac{1}{m(Q_j)} \int_{Q_j} |f| > t$$

and

$$|f| \leq t \text{ on } Q \setminus \bigcup_{j \geq 1} Q_j$$

Let $f \in L^1(\mathbb{R}^d)$. Then for every $t \in \mathbb{R}$ there are countably many disjoint dyadic cubes $\{Q_j\}$ in $\mathbb{R}^d$ such that

$$2^n t \geq \frac{1}{m(Q_j)} \int_{Q_j} |f| > t$$

and

$$|f| \leq t \text{ on } \mathbb{R}^d \setminus \bigcup_{j \geq 1} Q_j$$

(Calderson-Zygmund decomposition) Let $f \in L^1(\mathbb{R}^n)$ given $\lambda > 0$ there is a open Ω such that

$$f \leq \lambda \text{ ae on } \Omega^c$$

and

$$\Omega = \bigcup_k Q_k$$

where the dyadic cubes Q_k -s are almost disjoint mutually, and

$$\lambda < \frac{1}{m(Q_k)} \int_{Q_k} f \leq 2^n \lambda$$

decomposition into a sum of bounded and unbounded oscillatory *parts* with zero-mean

Let $f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. Then there exists a collection $\{Q_j\}$ of disjoint cubes such that

$$\sum_j m(Q_j) \leq \frac{1}{\alpha} \|f\|_1$$

and there exists $g \in L^1 \cap L^\infty(\mathbb{R}^n), b_j \in L^1(Q_j)$ such that

$$f \equiv g + \sum_j b_j$$

where

$$\begin{aligned} \|g\|_\infty &\leq 2^d \alpha \\ \|g\|_1 &\leq \|f\|_1 \\ g|_{Q_j} &\equiv \frac{1}{m(Q_j)} \int_{Q_j} f \end{aligned}$$

and

$$\begin{aligned} \int_{Q_j} b_j &= 0 \\ \frac{1}{m(Q_j)} \|b_j\|_1 &\leq 2^{d+1} \alpha \end{aligned}$$

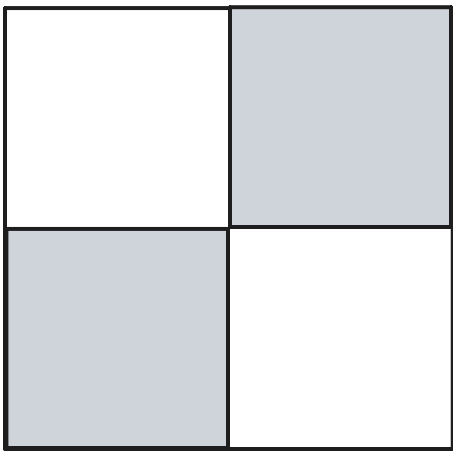
• Pick integer $N > 0$ such that

$$\|f\|_1 \leq \frac{\alpha}{2^{dN}}$$

• We divide $\mathbb{R}^d$ into Ω_N .

• Consider

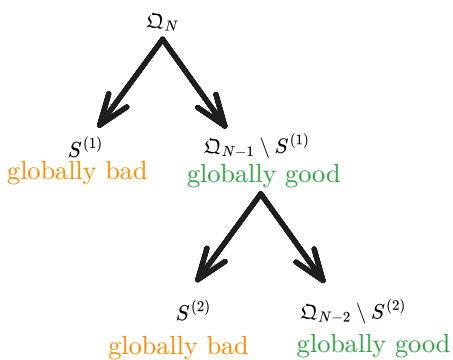
$$S^{(1)} := \left\{ Q \in \Omega_{N-1} \mid \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$



-  Cubes in $S^{(1)}$ are where f is **globally bad**. We consider all cubes in $\Omega_{N-1} \setminus S^{(1)}$ where f behaves **globally good**. We subdivide each into 2^n subcubes. Then we look at the ones where f is **globally bad**, and put them in $S^{(2)}$.

- Consider

$$S^{(2)} := \left\{ Q \in \Omega_{N-2}, Q \subseteq \bigcup \Omega_{N-1} \setminus S^{(1)} \mid \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$



- Continuing

$$S^{(k)} := \left\{ Q \in \Omega_{N-k}, Q \subseteq \bigcup \Omega_{N-k+1} \setminus S^{(k-1)} \mid \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$

- ...
- If $\bigcup_{m \geq 1} S^{(m)} = \emptyset$ then set $b \equiv 0$ and $g \equiv f$.
- Otherwise, we consider the countable collection of cubes

$$\{Q_j\} := \bigcup_{m \geq 1} S^{(m)}$$

which are all disjoint by construction.

-  **Definition.**

$$b_j := \left(f - \frac{1}{m(Q_j)} \int_{Q_j} f \right) 1_{Q_j}$$

-  **Definition.**

$$g := f 1_{\mathbb{R}^n \setminus \bigcup_j Q_j} + \sum_j \left(\frac{1}{m(Q_j)} \int_{Q_j} f \right) 1_{Q_j}$$

- By construction $f \equiv g + \sum_j b_j$.

Current note has 0 direct children and 0 total descendants.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \setminus \mathbb{R}^n, \Gamma \setminus \mathbb{C}^n\$](#)
 - [Lebesgue measurable subsets of and functions on \$\mathbb{R}^n, T^n, S^n\$](#)
 - [Calderon-Zygmund decomposition](#)

And it has 15 siblings.

- [stem](#)
 - [subobjects of and functions on \$\mathbb{R}^n, S^n, \mathbb{C}^n\$ and its quotients \$\Gamma \setminus \mathbb{R}^n, \Gamma \setminus \mathbb{C}^n\$](#)
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 - [\$L^{p,q}\$](#)
 - [\$E\(\[0,1\]\) = E\(0,1\), E\(\[- \pi, \pi\]\) = E\(S^1\)\$](#)