ⓘ Info

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Curves in smooth manifolds

Definition. Smooth curves in smooth manifolds

A continuous function

$$\gamma : I \to U \subseteq M$$

is called a **curve** in U.
A curve γ is smooth if for any chart (U, φ) the map

$$\varphi \circ \gamma : I \to \varphi(U)$$

is a **smooth curve** in $\mathbb{R}^{\dim M}$

Definition. Set of smooth curves

The set of all **smooth curves** is the set $\mathcal{C}^\infty(I, U)$ where $I \subseteq \mathbb{R}, U \subseteq M$. We define the set of all curves with the naturally desired domain,

$$\gamma : (-\epsilon, \epsilon) \to U \subseteq M$$

for some $\epsilon > 0$ as

$$\mathrm{Curves}^\infty_\epsilon(p \in U) := \{\gamma \in \mathcal{C}^\infty((-\epsilon, \epsilon), U) : \gamma(0) = p\}$$

velocity

equivalence classes of velocities

For a open set U on a n-manifold, and a chart (U, φ)

$$\mathrm{Curves}^\infty(p \in U) \to \mathbb{R}^n$$

such that

$$\gamma \mapsto (\varphi \circ \gamma)'(0)$$

this is a surjective map. Lets denote the map by the ugly $(\varphi \circ -)'(0)$. This map shall partition the domain

$$\mathrm{Curves}^\infty(p \in U)/(\varphi \circ -)'(0)$$

where the equivalence relation is

$$\gamma_1 \sim \gamma_2 \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

The $\mathrm{Curves}^\infty(p \in U)/(\varphi \circ -)'(0)$ partition does not depend on the chart φ.

Definition. The equivalence classes of velocity

Denote the quotient set of the equivalence classes of velocity by

$$\mathrm{Curves}^\infty(p \in U)/(-)'(0)$$

and the equivalence classes by

$$[\gamma'(0)]$$

because it does not depend on chart.

canonical bijection with T_pM

canonical bijection

We have

$$\begin{tikzcd}
T_pM \arrow[r, "(-\circ\gamma)'(0) \mapsto (\varphi\circ\gamma)'(0)"] & \mathbb{R}^{\dim M} \\
\mathrm{Curves}^\infty(p \in U) \arrow[ur, "\gamma \mapsto (\varphi \circ \gamma)'(0)"'] &
\end{tikzcd}$$

From

> For a n-manifold M, map
>
> $$T_pM \to \mathbb{R}^n$$
> $$(- \circ \gamma)'(0) \mapsto (\varphi \circ \gamma)'(0)$$
>
> is an $\mathbb{R}$-linear isomorphism for any local chart φ at p. Hence,
>
> $$\dim(T_pM) = \dim(M).$$

we know the operator $(- \circ \gamma)'(0) \in T_pM$ is **same** for two curves in $[\gamma'(0)]$ which is

$$(- \circ \gamma_1)'(0) = (- \circ \gamma_2)'(0) \iff (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

Definition. The **canonical bijection** $T_pM \to \mathrm{Curves}^\infty(p \in U)/(-)'(0)$ is defined as

$$(-\circ\gamma)'(0) \leftrightarrow [\gamma'(0)]$$

This the diagram

$$\begin{tikzcd} T_pM \arrow[r, "(-\circ\gamma)'(0) \mapsto (\varphi\circ\gamma)'(0)"] & \mathbb{R}^{\dim M} \\ \mathrm{Curves}^\infty(p \in U) \arrow[u, "(-\circ\gamma)'(0) \mapsfrom [\gamma'(0)]"] \arrow[ur, "\gamma \mapsto (\varphi\circ\gamma)'(0)"'] & \end{tikzcd}$$

commutes and each arrow is an isomorphism.

velocity

Definition. Velocity of a curve at a point

Given a smooth curve $\gamma : (-\epsilon, \epsilon) \to U$ the **velocity at** $\gamma(0) \in M$ is defined as

$$\begin{aligned} \gamma'(0) :=& (-\circ\gamma)'(0) \in T_pM \\ =& [\gamma'(0)] \qquad \in \mathrm{Curves}^\infty(\gamma(0) \in U)/(-)'(0) \end{aligned}$$

> $\gamma' : I \to T_{\gamma(t)}M$
>
> $$t \mapsto (-\circ\gamma)'(t)$$
>
> ~

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Curves in smooth manifolds

And it has 19 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Smooth Lie group action on a smooth manifold
      - Borel measures on a smooth manifold
      - G-bundles over smooth manifolds $G \curvearrowright P \to X$
      - Vector bundle on smooth manifolds
      - Curves in smooth manifolds
      - Density on smooth manifolds
      - Functions between smooth manifolds
      - Functions from smooth manifolds to $\mathbb{R}$
      - Flows on a smooth manifold
      - Locally $G \curvearrowright X$-manifolds
      - Smooth singular homology of a smooth manifold
      - Isomorphisms and automorphisms of smooth manifolds
      - Lagrangian flows
      - Local transformation rules
      - Moduli space of Riemannian metrics on a smooth manifold
      - Moduli space of quotients
      - Quotient manifolds
      - Submanifolds of smooth manifolds
      - Tangents on a smooth manifold