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Proper maps between topological spaces

Definition. Proper, perfect, universally closed maps

A continuous map $f : X \rightarrow Y$ between topological spaces is

- **proper** if preimage of compact sets $K \subseteq Y$, $f^{-1}(K)$ are compact in X
- **Bourbaki-proper** if for any topological space Z the product map

$$\text{Id}_Z \times f : Z \times X \rightarrow Z \times Y$$

is closed

- **perfect** if f is closed and has compact fibres ($f^{-1}(y)$ is compact for all $y \in Y$)
- **universally closed** if for every continuous map $g : Z \rightarrow Y$ the pullback

$$g \times_Y f : Z \times_Y X \rightarrow Z$$

is closed

- **sequential escaper** if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ *escaping to infinity* then $\{f(x_n)\}_{n \in \mathbb{N}}$ *escapes to infinity*

[1]

[2]

[3]

Let $f : X \rightarrow Y$ be a continuous map between topological spaces. Then **f is perfect $\implies f$ is proper.**

Let $K \subseteq Y$ be compact and $\mathcal{O} = \{U_\alpha \subseteq X\}$ be an open cover of $f^{-1}(K)$.

- For each $p \in K$, $f^{-1}(p)$ is compact and $\mathcal{O}$ covers it, so it has a finite subcover

$$\mathcal{O}_p \subseteq \mathcal{O}$$

- Let

$$U'_p := \bigcup \mathcal{O}_p \supseteq f^{-1}(p)$$

It is open in X . Thus $f(X \setminus U'_p)$ is closed in Y . So

$$Y \setminus f(X \setminus U'_p)$$

is open in Y .

$$\begin{aligned} q \in X \setminus U'_p \subseteq X \setminus f^{-1}(p) &\implies f(q) \neq p \\ &\implies p \notin f(X \setminus U'_p) \\ &\implies p \in Y \setminus f(X \setminus U'_p) \end{aligned}$$

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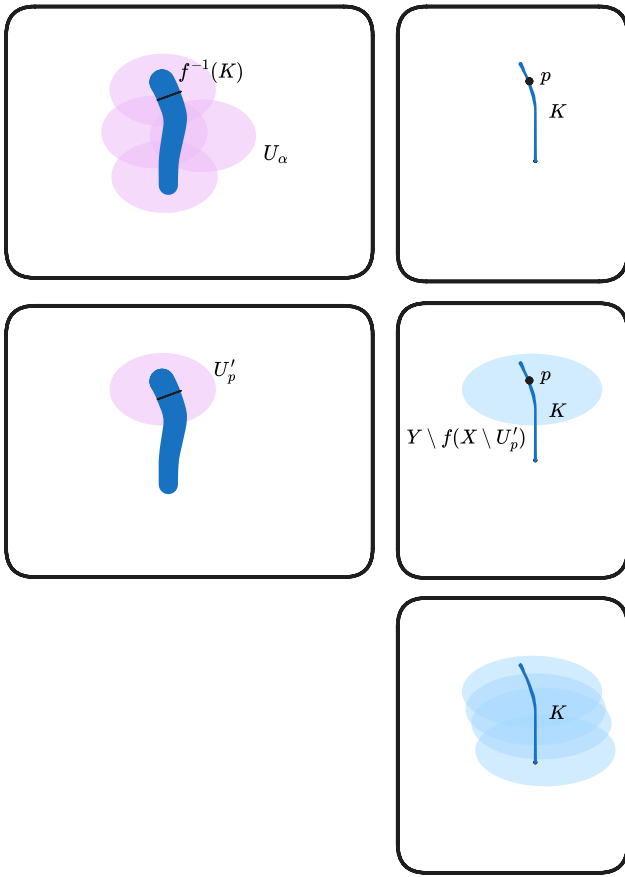
$$U'_{p_i} \subseteq Y \setminus f^{-1}(f(X \setminus U'_{p_i}))$$

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$$\{Y \setminus f(X \setminus U'_p) \mid p \in K\}$$

is an open cover of K , thus has a finite subcover

$$\{Y \setminus f(X \setminus U'_{p_i}) \mid 1 \leq i \leq N\}$$



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$$\begin{aligned} K &\subseteq \bigcup_i Y \setminus f(X \setminus U'_{p_i}) \\ f^{-1}K &\subseteq \bigcup_i f^{-1}(Y \setminus f(X \setminus U'_{p_i})) \\ &= \bigcup_i \underbrace{Y \setminus f^{-1}(f(X \setminus U'_{p_i}))}_{\subseteq U'_{p_i}} \\ &= \bigcup_i U'_{p_i} \end{aligned}$$

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which means

$$\bigcup_{i \leq N} \mathcal{O}_{p_i}$$

is a finite subcover of $\mathcal{O}$.

universally closed maps

[4]

when are proper maps closed

Proposition: Let $f : X \rightarrow Y$ be a proper map which is either

- between two locally compact Hausdorff spaces
- between two metric spaces

Then f is closed

[5]

proper maps whose images are not closed

📌 **Definition.** A proper map $f : X \rightarrow Y$ is called **bad-proper** if the image $f(X) \subset Y$ is not closed.

Proposition: Let $f : X \rightarrow Y$ be a bad-proper map. Then the inclusion

$$\iota : f(X) \hookrightarrow Y$$

is bad-proper.

Conversely, given a bad-proper inclusion $\iota : F \hookrightarrow Y$ and a continuous, proper surjection $X \twoheadrightarrow F$ gives a bad-proper

$$X \rightarrow Y$$

📌 **Definition.** A subset $F \subset X$ is **bad-proper** if the inclusion map $\iota : F \hookrightarrow X$ is bad-proper, that is, it is a proper map and $F \subset X$ is not closed.

[6]

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1. [Section 5.17 \(005M\): Characterizing proper maps—The Stacks project](#) ↩↗
 2. [general topology - Definitions of proper maps - Mathematics Stack Exchange](#) ↩↗
 3. [proper.pdf](#) ↩↗
 4. [compactness-equivalence-and-application-to-proper-maps.pdf](#) ↩↗
 5. [Proper Maps.pdf](#) ↩↗
 6. [proper_image_2.pdf](#) ↩↗