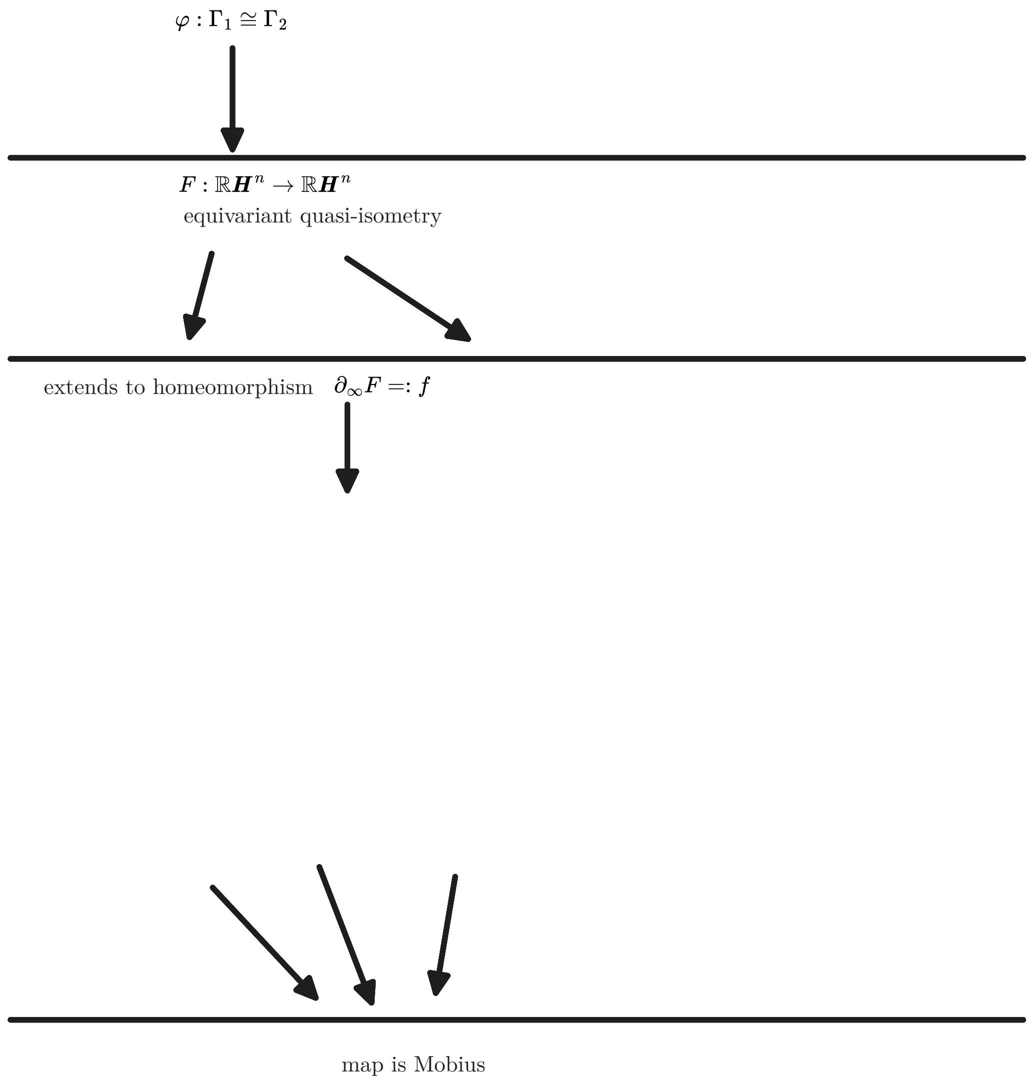
Rigidity of $\mathbb{R}$-hyperbolic manifolds

(Mostow rigidity) Let M, N be **compact** $\mathbb{R}$-hyperbolic n-manifolds for $n \geq 3$ with

$$\phi : \pi_1(M, \bullet) \cong \pi_1(N, \blacksquare)$$

Then there exists an unique isometry $f : M \to N$ such that $f_* = \phi$.



(Mostow)
(Gromov and Thurston) Rigidity using measure homology
(Besson, Courtois, and Gallot) Volume entropy
(Sullivan) Using ergodic current on $\partial^2\mathbb{R}\boldsymbol{H}^n$
(Sullivan, Tukia) Rigidity of lattices using quasi-conformal mappings

Proposition: Let M, N be **compact** $\mathbb{R}$-hyperbolic n-manifolds for $n \geq 3$ with

$$\phi : \pi_1(M, m) \cong \pi_1(N, n)$$

Then

- it induces a homotopy equivalence

$$h : M \to N$$

- which lifts to a quasi-isometry between their universal covers

$$\tilde{h} : \mathbb{R}\boldsymbol{H}^n \to \mathbb{R}\boldsymbol{H}^n$$

- This extends to a homeomorphism of the boundary

$$\partial\tilde{h} : S^{n-1} \to S^{n-1}$$

which is π_1-equivariant $\partial\tilde{f} \circ \gamma = \phi(\gamma) \circ \partial\tilde{f}$ for all $\gamma \in \pi_1(M, m)$.

lifting isomorphism of lattices to the boundary

PROOF. **Step 1.** We first observe that Γ is uniform if and only if Γ' is uniform. Indeed, if Γ is uniform, it is Gromov-hyperbolic and, hence, cannot contain a non-cyclic free abelian group. On the other hand, if Γ' is non-uniform, then Corollary 12.28 implies that Γ' contains free abelian subgroups of rank $n-1>1$.

LEMMA 24.16. *There exists a ρ-equivariant quasiisometry $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$.*

PROOF. As in the proof of Theorem 24.1, we choose truncated hyperbolic spaces $\Omega \subset \mathbb{H}^n, \Omega' \subset \mathbb{H}^n$ for the lattices Γ and Γ' respectively. (If Γ is a uniform lattice, we take, of course, $\Omega = \Omega' = \mathbb{H}^n$.) Lemma 8.45 implies that there exists a ρ-quasiequivariant quasiisometry

$$f : \Omega \to \Omega'.$$

Therefore, according to Theorem 24.14, f extends to a ρ-equivariant quasiisometry $\tilde{f} : \mathbb{H}^n \to \mathbb{H}^n$. □

REMARK 24.17. The most difficult part of the proof of Theorem 24.14 was to show that f sends peripheral horospheres uniformly close to peripheral horospheres. In the equivariant setting the proof is much easier: The homomorphism ρ sends maximal abelian subgroups of Γ to maximal abelian subgroups of Γ'. The stabilizers of peripheral horospheres are virtually $\mathbb{Z}^{n-1}$. Therefore, ρ sends stabilizers of peripheral horospheres to stabilizers of peripheral horospheres. From this, it is immediate that peripheral horospheres map uniformly close to peripheral horospheres.

Current note has 3 direct children and 3 total descendants.

- stem

- O+ R1 n
    - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
        - Rigidity of $\mathbb{R}$-hyperbolic manifolds
            - Using ergodic current on $\partial^2\mathbb{R}\boldsymbol{H}^n$
            - Rigidity using measure homology
            - Rigidity of lattices using quasi-conformal mappings

And it has 11 siblings.

- stem
    - O+ R1 n
        - Discrete subgroups of $O^+(1,n)$ $\leftrightarrow$ hyperbolic n-manifolds/orbifolds
            - Convex cocompact discrete subgroups
            - Combination of discrete subgroups of $O^+(1,n)(\mathbb{R})$
            - Exponential sum of a discrete subgroup of $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n)$
            - $\mathrm{RepDF}(F_g, O^+(1,n)(\mathbb{R}))$
            - Measure homology on a hyperbolic manifold
            - Discrete subgroups with limit set a discontinuum
            - Limit sets of discrete subgroups
            - Discrete subgroups with limit set a $k-1$-sphere
            - Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}\boldsymbol{H}^n$
            - Rigidity of $\mathbb{R}$-hyperbolic manifolds
            - $\mathrm{RepDF}(\pi_1(T_g^2, \bullet), O^+(1,n)(\mathbb{R}))$