ⓘ Info

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Global holomorphic functions on a complex 1-manifold

7.1. Definition. Suppose X is a Riemann surface, $u\colon [0,1] \to X$ is a curve and $a := u(0)$, $b := u(1)$. The holomorphic function germ $\psi \in \mathcal{O}_b$ is said to result from the *analytic continuation along the curve* u of the holomorphic function germ $\varphi \in \mathcal{O}_a$ if the following holds. There exists a family $\varphi_t \in \mathcal{O}_{u(t)}$, $t \in [0,1]$ of function germs with $\varphi_0 = \varphi$ and $\varphi_1 = \psi$ with the property that for every $\tau \in [0,1]$ there exists a neighborhood $T \subset [0,1]$ of τ, an open set $U \subset X$ with $u(T) \subset U$ and a function $f \in \mathcal{O}(U)$ such that

$$\rho_{u(t)}(f) = \varphi_t \quad \text{for every } t \in T.$$

Here $\rho_{u(t)}(f)$ is the germ of f at the point $u(t)$. Because of the compactness of $[0,1]$ this condition is equivalent to the following (see Fig. 5). There exist a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of the interval $[0,1]$, domains $U_i \subset X$ with $u([t_{i-1}, t_i]) \subset U_i$ and holomorphic functions $f_i \in \mathcal{O}(U_i)$ for $i = 1, \ldots, n$ such that:

(i) φ is the germ of f_1 at the point a and ψ is the germ of f_n at the point b.
(ii) $f_i \mid V_i = f_{i+1} \mid V_i$ for $i = 1, \ldots, n-1$, where V_i denotes the connected component of $U_i \cap U_{i+1}$ containing the point $u(t_i)$.

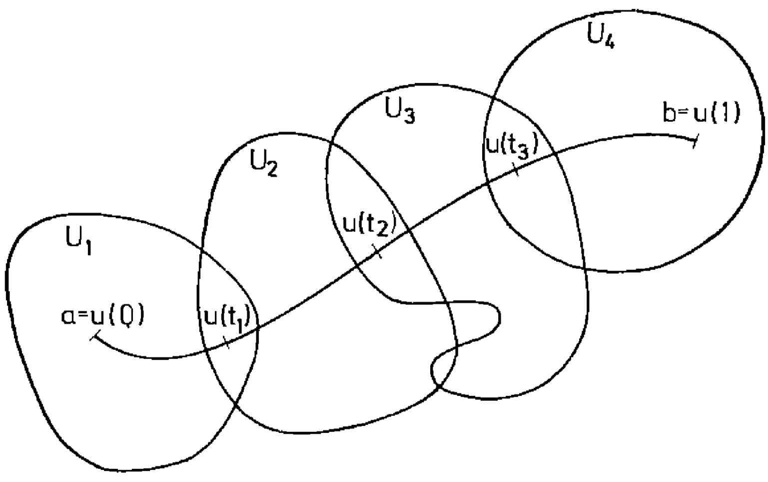

Figure 5

If one carries out the construction given in (6.7) for the sheaf $\mathcal{O}$ of holomorphic functions, then one gets a map $p\colon |\mathcal{O}| \to X$. The next Lemma shows that one can interpret analytic continuation along a curve by means of this map.

7.2. Lemma. *Suppose X is a Riemann surface and $u\colon [0,1] \to X$ is a curve in X with $u(0) =: a$ and $u(1) =: b$. Then a function germ $\psi \in \mathcal{O}_b$ is the analytic continuation of a function germ $\varphi \in \mathcal{O}_a$ along u precisely if there exists a lifting $\hat{u}\colon [0,1] \to |\mathcal{O}|$ of the curve u such that $\hat{u}(0) = \varphi$ and $\hat{u}(1) = \psi$.*

PROOF
(a) Suppose $\psi \in \mathcal{O}_b$ is the analytic continuation of $\varphi \in \mathcal{O}_a$ along u. Let $\varphi_t \in \mathcal{O}_{u(t)}$ for $t \in [0,1]$ be the family of function germs as given in the Definition (7.1). It follows directly from the definition of the topology of $|\mathcal{O}|$ that the correspondence $t \mapsto \varphi_t$ represents a continuous mapping $\hat{u}\colon [0,1] \to |\mathcal{O}|$. Thus $\hat{u}$ is a lifting of u with $\hat{u}(0) = \varphi_0 = \varphi$ and $\hat{u}(1) = \varphi_1 = \psi$.
(b) Suppose there is a lifting $\hat{u}\colon [0,1] \to \mathcal{O}$ of u with $\hat{u}(0) = \varphi$ and $\hat{u}(1) = \psi$. For $t \in [0,1]$, let $\varphi_t := \hat{u}(t)$. Then $\varphi_t \in \mathcal{O}_{u(t)}$ and $\varphi_0 = \varphi$, $\varphi_1 = \psi$. Let $\tau \in [0,1]$ and suppose $[U, f] \subset |\mathcal{O}|$ is an open neighborhood of $\hat{u}(\tau)$. Then there exists a neighborhood $T \subset [0,1]$ of τ such that $\hat{u}(T) \subset [U, f]$. This implies $u(T) \subset U$ and $\varphi_t = \hat{u}(t) = \rho_{u(t)}(f)$ for every $t \in T$. But this means that ψ is the analytic continuation of φ along u. □

global holomorphic functions are maximum analytic continuations

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Definition. Analytic continuations of holomorphic germs

Let X be a Riemann surface, $a \in X$ and $f_a \in \mathcal{O}_a$ is a germ. Then a *pointed unbranched holomorphic map* ψ and a holomorphic f

$$\begin{array}{ccc} (Y,b) & \xrightarrow{f} & \mathbb{C} \\ \big\downarrow{\scriptstyle\psi} & & \\ (X,a) & & \end{array}$$

is called an **analytic continuation** of f_a if

$$p_*(\rho_b(f)) = f_a$$

Definition. Maximal analytic continuations of holomorphic germs

Let X be a Riemann surface. An analytic continuation $(Y,b) \to (X,a)$ of $f_a \in \mathcal{O}_a$ is said to be **maximal** if: for any other analytic continuation

$$\begin{tikzcd}
(Z,z) \arrow[rr, bend left, "g"] \arrow[dr, "\phi"'] & (Y,b) \arrow[r, "f"] \arrow[d, "\psi"] & \mathbb{C} \\
& (X,a) &
\end{tikzcd}$$

there exists holomorphic

$$F : Z \to Y$$

such that

$$\begin{tikzcd}
(Z,z) \arrow[rr, bend left, "g"] \arrow[r, "F"] \arrow[dr, "\phi"'] & (Y,b) \arrow[r, "f"] \arrow[d, "\psi"] & \mathbb{C} \\
& (X,a) &
\end{tikzcd}$$

> **Proposition:** A maximal analytic continuation is unique up to isomorphism.

PROOF. For $t \in [0, 1]$ let $\varphi_t := p_*(\rho_{v(t)}(f)) \in \mathcal{O}_{p(v(t))} = \mathcal{O}_{u(t)}$. Then $\varphi_0 = \varphi$ and $\varphi_1 = p_*(f_y) = \psi$. Suppose $t_0 \in [0, 1]$. Since $p\colon Y \to X$ is a local homeomorphism, there exist open neighborhoods $V \subset Y$ and $U \subset X$ of $v(t_0)$ and $p(v(t_0)) = u(t_0)$ resp. such that $p \,|\, V \to U$ is biholomorphic. Let $q\colon U \to V$ be the inverse mapping and let $g := q^*(f \,|\, V) \in \mathcal{O}(U)$. Then $p_*(\rho_\eta(f)) = \rho_{p(\eta)}(g)$ for every $\eta \in V$. There exists a neighborhood T of t_0 in $[0, 1]$ such that $v(T) \subset V$, i.e., $u(T) \subset U$. For every $t \in T$

$$\rho_{u(t)}(g) = p_*(\rho_{v(t)}(f)) = \varphi_t .$$

This proves that ψ is an analytic continuation of φ along u. $\square$

Let X be a Riemann surface, $a \in X$ and $f_a \in \mathcal{O}_a$. Then there **exists** a *maximal analytic continuation* of f_a: the connected component of f_a in $|\mathcal{O}_X|$.

Current note has 0 direct children and 0 total descendants.

And it has 3 siblings.