

 Info

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$$H^1(U)$$

 Definition. Sobolev space $H^1(U)$

Let $U(\text{open}) \subseteq \mathbb{R}^n$ be an open set. Then the **Sobolev space** of $L^2(U)$ functions with [distributional derivatives](#) in $L^2(U)$

$$H^1(U) := \left\{ f \in L^2(U) \middle| \forall 1 \leq j \leq n, \int \partial_j f \in L^2(U) \right\}$$

is equipped with the inner product

$$\langle f, g \rangle_{H^1} := \langle f, g \rangle_2 + \sum_{1 \leq j \leq n} \left\langle \int \partial_j f, \int \partial_j g \right\rangle_2$$



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is a separable Hilbert space such that

$$\begin{aligned} \text{Id} \oplus \text{grad} : H^1(U) &\rightarrow \bigoplus_{j=0}^n L^2(U) \\ f &\mapsto (f, \int \partial_1 f, \dots, \int \partial_n f) \end{aligned}$$

is a linear isometric embedding.

[1]

 Every function in $H^1(a, b)$ has a representative in $\mathcal{C}[a, b]$ and the inclusion

$$H^1(a, b) \hookrightarrow \mathcal{C}[a, b]$$

is continuous.

 As

$$f(x) = \int_{[a, x]} f'$$

we have a continuous representative.

- Now

$$|f(x)| \leq |f(a)| + \underbrace{\int_{[a, x]} |f'|}_{\leq \int_{[a, b]} |f'|}$$

$$\begin{aligned} \|f\|_\infty &\leq \inf_{[a, b]} |f| + \int_{[a, b]} |f'| \\ &\leq \frac{1}{b-a} \int_{(a, b)} |f| + \int_{[a, b]} |f'| \\ &\leq \sqrt{b-a} \|f\|_2 + \sqrt{b-a} \|f'\|_2 \end{aligned}$$

 Definition.

$$H_0^1(U) := \overline{\{f \in H^1(U) \mid \text{supp } f \subset_{\text{cpt}} U\}}^{H^1(U)}$$

Therefore $H_0^1(U)$ is a Hilbert space as well.

 Intuition

$$H_0^1(U) = \{f \in H^1(U) \mid f|_{\partial U} \equiv 0\}$$

equivalent inner product

By Poincare inequality

$$\|\text{grad}(u)\|_2^2 \leq \|u\|_{H^1(U)}^2 \leq C \|\text{grad}(u)\|_2^2$$

 Let $u \in H^1(U)$ and $g \in L^2(U)$. Then

$$\begin{aligned} \Delta u &= g \\ \iff - \sum_{j=1}^n \left\langle \int \partial_j -, \int \partial_j u \right\rangle_{L^2(U)} &= \langle -, g \rangle_{L^2(U)} \text{ on } H_0^1(U) \\ \iff - \langle -, u \rangle_{H^1(U)} &\equiv \langle -, g - u \rangle_{L^2} \text{ on } H_0^1(U) \end{aligned}$$

- Let $f \in L^2(U)$. Then we have the functional

$$\langle -, f \rangle_2$$

on $L^2(U)$.

- It is continuous on $H_0^1(U)$ because

$$\langle -, f \rangle_2 \leq \|f\|_2 \underbrace{\|-\|_2}_{\leq \|-\|_{H^1(U)}}$$

- By Riesz representation theorem there exists a $u \in H_0^1(U)$ such that

$$\langle -, f \rangle_2 \equiv \langle -, u \rangle_{H^1} \text{ on } H_0^1(U)$$

- Therefore,

$$\int \Delta u = u - f$$

- Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^2(\Omega)$. Then there exists an unique $H_0^1(U)$ such that

$$\int \Delta u = u - f$$

- Let $f \in L^2(U)$. Then we have the functional

$$\langle -, f \rangle_2$$

on $L^2(U)$.

- It is continuous on $H_0^1(U)$ because

$$\langle -, f \rangle_2 \leq \|f\|_2 \underbrace{\|-\|_2}_{\leq \|-\|_{H^1(U)}}$$

- Then by considering the inner product $\langle \text{grad}(-), \text{grad}(-) \rangle_2$, by *Riesz representation* we have an unique $u \in H_0^1(U)$ such that

$$\langle \text{grad}(-), \text{grad}(u) \rangle_2 \equiv \langle -, f \rangle_2 \text{ on } H_0^1(U)$$

- Therefore,

$$-\int \Delta u = f$$

Current note has 0 direct children and 0 total descendants.

- stem
 - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - Differentiable functions
 - Distributional derivatives
 - $H^1(U)$

And it has 4 siblings.

- stem
 - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
 - Differentiable functions
 - Distributional derivatives
 - Fractional Sobolev spaces
 - $H^1(U)$
 - H^1_{loc}
 - $H^1_{\text{loc}}(U)$ for $U \subseteq \mathbb{R}^2$