Info

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Meromorphic functions on a Riemann surface

algebraic degree and degree of meromorphic maps

Let Σ be a compact Riemann surface. It admits a non-trivial holomorphic

$$f : \Sigma \to \mathbb{C}P^1$$

Then

$$\begin{aligned} f_* : \mathbb{C}[\mathbb{C}P^1] \cong \mathbb{C}(z) &\hookrightarrow \mathbb{C}(\Sigma) \\ p(z) &\mapsto p(f(-)) \end{aligned}$$

is an embedding of fields.

$$[\mathbb{C}(\Sigma), \mathbb{C}(z)] = \deg f$$

Let (by translation if necessary) $0 \in \mathbb{C}P^1$ is a regular value of f so $f^{-1}(0)$ is a set of $d := \deg f$ distinct points

$$p_1, \ldots, p_d \in \Sigma$$

- By *existence* for each i there is a meromorphic

  $$g_i : \Sigma \to \mathbb{C}P^1$$

  that has a simple pole at p_i but is holomorphic around p_j for each $j \neq i$.
- Suppose

  $$\sum_j \lambda_j g_j = 0$$

  where $\lambda_j \in \mathbb{C}(z) \hookrightarrow \mathbb{C}(\Sigma)$. By multiplying a suitable z^k we can assume λ_j are holomorphic around $z = 0$ and do not all vanish at $z = 0$.
- Consider

  $$\sum_j \lambda_j(f(p_i)) g_j(p_i) = 0$$

  The only non-holomorphic term is $\lambda_i g_i$ so

  $$\lambda_j(\underbrace{f(p_i)}_{0}) = 0$$

  which is a contradiction.
- Thus

  $$[\mathbb{C}(\Sigma), \mathbb{C}(z)] \geq \deg f$$

- By *primitive element theorem* it suffices to show any $g \in \mathbb{C}(\Sigma)$ satisfies

  $$P(f, g) = \lambda_0(f) + g\lambda_1(f) + g^2\lambda_2(f) + \cdots + g^d\lambda_d(f)$$

  - Let z be a regular value of f so

    $$f^{-1}(z) = \{p_1, \ldots, p_d\} \subset \Sigma$$

    such that

    $$w_i := g(p_i) \in \mathbb{C} \subset \mathbb{C}P^1$$

  - If

    $$a_1 := \sum_i w_i$$

    and so on are elementary symmetric functions we have

    $$\sum_{r=0}^{d} (-1)^r a_r g^r = 0$$

holomorphic maps and field extensions

> **Corollary of**
>
> Let Σ be a compact Riemann surface. It admits a non-trivial holomorphic
>
> $$f : \Sigma \to \mathbb{C}P^1$$
>
> Then
>
> $$\begin{aligned} f_* : \mathbb{C}[\mathbb{C}P^1] \cong \mathbb{C}(z) &\hookrightarrow \mathbb{C}(\Sigma) \\ p(z) &\mapsto p(f(-)) \end{aligned}$$
>
> is an embedding of fields.
>
> $$[\mathbb{C}(\Sigma), \mathbb{C}(z)] = \deg f$$
>
> If
>
> $$f : \Sigma_1 \to \Sigma_2$$
>
> is a non-constant holomorphic map between compact connected Riemann surfaces then
>
> $$[\mathbb{C}(\Sigma_1), \mathbb{C}(\Sigma_2)] = \deg f$$

Consider

$$f_0 : \Sigma_2 \to \mathbb{C}P^1$$

then

$$\mathbb{C}(z) \hookrightarrow \mathbb{C}(\Sigma_2) \hookrightarrow \mathbb{C}(\Sigma_1)$$

which gives

$$[\mathbb{C}(\Sigma_1), \mathbb{C}(z)] = [\mathbb{C}(\Sigma_1), \mathbb{C}(\Sigma_2)][\mathbb{C}(\Sigma_2), \mathbb{C}(z)]$$

Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $\mathbb{R}$-spaces of dimension 2
    - Riemann surfaces
      - Meromorphic functions on a Riemann surface

And it has 16 siblings.

- stem
  - Objects stemming from $\mathbb{R}$-spaces of dimension 2
    - Riemann surfaces
      - Vector bundles on complex 1-manifolds
      - Classification of complex 1-manifold
      - Riemann's existence of holomorphic maps between compact Riemann surfaces
      - Divisors on compact Riemann surfaces
      - Fluid
      - Meromorphic differentials on Riemann surfaces
      - Holomorphic functions on a complex 1-manifold
      - Holomorphic maps of type $0 \xrightarrow{3} 0$
      - Rational maps
      - Hurwitz scheme
      - Moduli space of $\mathbb{C}$-manifolds
      - Meromorphic functions on a Riemann surface
      - Normalization of plane projective $\mathbb{C}$-curves and Riemann surfaces
      - Riemann surfaces defined over a number field
      - Hyperelliptic Riemann surfaces
      - Riemann surfaces with a holomorphic 1-form