Info

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Homogeneous manifold

Definition. Homogeneous manifold

A **transitive** smooth Lie group action on a smooth $\mathbb{R}$-manifold is called a homogeneous G-manifold.

Let $\theta : G \curvearrowright M$ is a homogeneous G-manifold. Then the orbit map

$$\theta : {}^{G}/_{G_p} \cong_{\mathsf{Man}_G} M$$

is a G-isomorphism.

[1]

The derivative of the orbit map

$$\mathfrak{D}_i\theta : \underbrace{T_i\left({}^{G}/_{G_p}\right)}_{\mathfrak{g}/\mathfrak{g}_p} \to T_pM$$

Is every smooth manifold homogeneous?

- Given any two points in a connected manifold, there exists a diffeomorphism taking one to the other. [1]
- However, not all smooth manifolds are homogeneous G-spaces for (finite dim) Lie groups G, there are necessary conditions [2]

(Mostow, 2005) M is a compact homogeneous G-space $\implies \chi(M) \geq 0$ [3]

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1. differential geometry - Proving that given any two points in a connected manifold, there exists a diffeomorphism taking one to the other - Mathematics Stack Exchange ↩
2. dg.differential geometry - Example of a manifold which is not a homogeneous space of any Lie group - MathOverflow ↩
3. A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata ↩

Is a quotient of a Lie-homogeneous again Lie-homogeneous?

No. By

> **(Mostow, 2005)** M is a compact homogeneous G-space $\implies \chi(M) \geq 0$ [1]

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1. A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata ↩

higher genus surfaces are not Lie-homogeneous, but their universal cover is $\mathbb{R}^2$ which is Lie-homogeneous.

The fact that they are not Riemannian homogeneous is easily proven by Riemann surface techniques. [1]

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1. https://mathoverflow.net/a/104106 ↩

Current note has 8 direct children and 8 total descendants.

- structures based on sets
  - manifolds
    - Homogeneous manifold
      - Analytic geometries
      - Invariant Riemannian metrics on homogeneous manifolds
      - Klein geometry
      - Nilmanifolds
      - Reductive homogeneous spaces
      - Naturally reductive homogeneous spaces
      - Homogeneous manifold with compact stabilizer
      - Thurston geometries

And it has 20 siblings.

- structures based on sets
  - manifolds
    - Closed aspherical manifolds
    - $\mathbb{C}$-manifold
    - Hermitian manifolds
    - Kahler manifolds
    - Kahler manifolds of complex dimension 2, AKA Kahler surfaces
    - smooth $\mathbb{R}$-manifold
    - Principal bundles over smooth manifolds
    - Manifold with a flow
    - $\mathbb{R}$-manifold group (Lie groups)
    - Homogeneous manifold
    - Finsler manifold
    - $\mathbb{R}$-manifold semi-Riemannian
    - Sub-Riemannian manifolds
    - $\mathbb{R}$-manifold Poisson
    - $\mathbb{R}$-manifold symplectic
    - Multisymplectic manifolds
    - Spin $\mathbb{R}$-manifold
    - $\mathbb{R}$-manifold topological
    - Toric manifolds
    - Manifolds with compatible triples

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1. mathi.uni-heidelberg.de/~lee/MenelaosSS16.pdf ↩