

This note, that [was found here](#), as a part of [a collection](#) is written (completely with human hands) by [Rupadarshi Ray](#), was created on April 9, 2024 2:51:45 PM, was last modified on September 7, 2026 1:11:14 PM, and was exported on September 7, 2026 3:09:58 PM.

Reconstructing meromorphic functions from stalks



(Holomorphic function on open subsets of $\mathbb{CP}^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{CP}^1$ and $A \subseteq \Omega$ with no limit point in Ω . For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \rightarrow \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a .

[Ash and Novinger - Complex Variables, p.144](#)

For such a f , the reciprocal

$$\frac{1}{f} : \Omega \setminus A \rightarrow \mathbb{C}$$

will be holomorphic with poles of order m_a precisely at every $a \in A$.

reconstructing meromorphic function from its stalks at poles



(Mittag-Leffler's theorem for open sets on $\mathbb{C}$) Let $U \subseteq \mathbb{C}$ be open and $E \subseteq U$ with its limit points in ∂U . For $a \in E$ let p_a be a polynomial with $p_a(0) = 0$. Then there exists a unique meromorphic function up to addition of a holomorphic function

$$f \in \frac{\text{Mer}(U)}{\text{Hol}(U)}$$

whose poles are precisely the elements of E and for each such pole $a \in E$, the function

$$f(z) - p_a\left(\frac{1}{z-a}\right)$$

has a *removable singularity* at a , that is the principal part of f at a is $p_a\left(\frac{1}{z-a}\right)$.

If the "principal part", which are elements of the quotient of stalks at $a \in E$

$$p_a\left(\frac{1}{z-a}\right) \in \frac{\text{Mer}_a}{\text{Hol}_a}$$

is given, then there is an unique

$$f \in \frac{\text{Mer}(U)}{\text{Hol}(U)}$$

which has these principal parts.

▲ How nice is this give and take? Claim: Fix such a discrete $E \subseteq U$ then

$$\text{Mitt} : \bigoplus_{a \in E} \frac{\text{Mer}_a}{\text{Hol}_a} \rightarrow \frac{\text{Mer}(U)}{\text{Hol}(U)}$$

is a linear map?, whose image is the space of all maps whose poles are exactly the elements of E . The inverse of this map is just mapping a meromorphic f on U to its stalk at $a \in U$. The first one is just polynomials in $\frac{1}{z-a}$

$$\bigoplus_{a \in E} \mathbb{C} \left[\frac{1}{z-a} \right]$$



reconstructing holomorphic functions from first few terms of its Taylor series at a discrete set of points



Let Ω be a open subset of $\mathbb{C}$ and $B \subseteq \Omega$ with no limit points in Ω . For each $b \in B$ let $P_b \in \mathbb{C}[z]$ be a polynomial. Then there is a holomorphic function

$$f : \Omega \rightarrow \mathbb{C}$$

such that the *truncated (first few terms of the) Taylor series* of f at b matches the polynomial P_b , that is

$$\sum_{k \leq \deg P_b} \frac{f^{(k)}(b)(z-b)^k}{k!} = P_b(z-b)$$

for each $b \in B$.

🔦 Apply

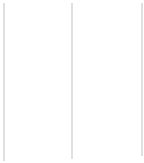


(Holomorphic function on open subsets of $\mathbb{CP}^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{CP}^1$ and $A \subseteq \Omega$ with no limit point in Ω . For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \rightarrow \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a .

[Ash and Novinger - Complex Variables, p.144](#)



- $\mathcal{O} \cap L^p$
- $\mathcal{O}(S^1)$
- Zeros and singularities of holomorphic functions