Info

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Reconstructing meromorphic functions from stalks

(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.

Ash and Novinger - Complex Variables , p.144

For such a f, the reciprocal

$$\frac{1}{f} : \Omega \setminus A \to \mathbb{C}$$

will be holomorphic with poles of order m_a precisely at every $a \in A$.

reconstructing meromorphic function from its stalks at poles

(Mittag-Leffler's theorem for open sets on $\mathbb{C}$) Let $U \subseteq \mathbb{C}$ be open and $E \subseteq U$ **with its limit points in ∂U**. For $a \in E$ let p_a be a polynomial with $p_a(0) = 0$. Then there exists a unique meromorphic function up to addition of a holomorphic function

$$f \in \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

whose poles are precisely the elements of E and for each such pole $a \in E$, the function

$$f(z) - p_a\left(\frac{1}{z-a}\right)$$

has a *removable singularity* at a, that is the principal part of f at a is $p_a\left(\frac{1}{z-a}\right)$.

If the "principal part", which are elements of the quotient of stalks at $a \in E$

$$p_a\left(\frac{1}{z-a}\right) \in \frac{\mathrm{Mer}_a}{\mathrm{Hol}_a}$$

is given, then there is an unique

$$f \in \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

which has these principal parts.

How nice is this give and take? Claim: Fix such a discrete $E \subseteq U$ then

$$\mathrm{Mitt} : \bigoplus_{a \in E} \frac{\mathrm{Mer}_a}{\mathrm{Hol}_a} \to \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

is a linear map?, whose image is the space of all maps whose poles are exactly the elements of E. The inverse of this map is just mapping a meromorphic f on U to its stalk at $a \in U$. The first one is just polynomials in $\frac{1}{z-a}$

$$\bigoplus_{a \in E} \mathbb{C}\left[\frac{1}{z-a}\right]$$

reconstructing holomorphic functions from first few terms of its Taylor series at a discrete set of points

Let Ω be a open subset of $\mathbb{C}$ and $B \subseteq \Omega$ with no limit points in Ω. For each $b \in B$ let $P_b \in \mathbb{C}[z]$ be a polynomial. Then there is a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that the *truncated (first few terms of the) Taylor series* of f at b matches the polynomial P_b, that is

$$\sum_{k \leq \deg P_b} \frac{f^{(k)}(b)(z-b)^k}{k!} = P_b(z-b)$$

for each $b \in B$.

Apply

(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.

Ash and Novinger - Complex Variables , p.144

to produce a holomorphic function $g:\Omega\to\mathbb{C}$ such that $f^{-1}(0)=B$ and for each $b\in B$ the multiplicity of f at b is $m_b:=\deg(P_b)+1$.

- Let $m\geq 1$ and

$$g(z)=\sum_{k\geq m}a_k(z-b)^k$$

be holomorphic at b and

$$R(z)=\frac{c_1}{z-b}+\dots+\frac{c_m}{(z-b)^m}$$

be a rational function. Then gR has a removable singularity at b so

$$gR(z)=d_0+d_1(z-b)+d_2(z-b)^2+\dots$$

where

$$\begin{aligned}d_0&=a_0c_m\\ d_1&=a_0c_{m-1}+a_1c_m\end{aligned}$$

  - For given $c_1,c_2\dots,c_m$ we may determine $d_0,d_1,\dots,d_{m-1}$.
  - Conversely, if g is given as above (order of zero at b is ≥ 1) and $d_0,\dots d_{m-1}$ are given complex number, because $a_0\neq 0$ we may solve the equations

$$\begin{aligned}c_m&=\frac{d_0}{a_0}\\ c_{m-1}&=\frac{d_1}{a_0}-\frac{a_1}{a_0}c_m\end{aligned}$$

inductively, to obtain, in order, $c_m,c_{m-1},\dots,c_1$. Thus we constructed R such that

$$(gR)(z)=\underbrace{d_0+d_1(z-b)+\dots+d_{m-1}(z-b)^{m-1}}_{\text{given}}+\text{higher order terms }\dots$$

- Hence, we have rational functions R_b for each $b\in B$ such that

$$g(z)R_b(z)=P_b(z)+\text{higher order terms }\dots$$

near b.
- Finally, by

> **(Mittag-Leffler's theorem for open sets on $\mathbb{C}$)** Let $U\subseteq\mathbb{C}$ be open and $E\subseteq U$ **with its limit points in** ∂U. For $a\in E$ let p_a be a polynomial with $p_a(0)=0$. Then there exists a unique meromorphic function up to addition of a holomorphic function
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> $$f\in\frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$
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> whose poles are precisely the elements of E and for each such pole $a\in E$, the function
>
> $$f(z)-p_a\left(\frac{1}{z-a}\right)$$
>
> has a *removable singularity* at a, that is the principal part of f at a is $p_a\left(\frac{1}{z-a}\right)$.

we obtain h, meromorphic on Ω, such that for each b

$$h-R_b$$

has a *removable singularity* at each b.
- Now around each b,

$$\begin{aligned}gh&=\underbrace{g(h-R_b)}_{\text{hol}}+\underbrace{gR_b}_{\text{hol}}\\ gh(z)&=(z^{\deg P_b+1}+\dots)+P_b(z)+\text{higher order terms }\dots\end{aligned}$$

Hence, gh extends to a holomorphic function on Ω which has the required properties.

[1]

- $\mathscr{O} \cap L^p$
- $\mathscr{O}(S^1)$
- Zeros and singularities of holomorphic functions

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1. Wayback Machine 6.3.3 ↩