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# Closed subsets of $\mathbb{R}^n$ equipped with smooth functions

**Definition.** Smooth closed subsets of $\mathbb{R}^n$

Let $X \subseteq \mathbb{R}^n$ be a closed subset. Consider the $\mathbb{R}$-algebra

$$\mathcal{C}^\infty(X) := \{f|_X | f \in \mathcal{C}^\infty(\mathbb{R}^n)\} \subseteq \mathcal{C}(X)$$

We call $(X, \mathcal{C}^\infty(X))$ a **smooth closed subset** of $\mathbb{R}^n$.

We have the **surjective** restriction

$$\mathrm{res} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(X)$$

whose kernel is $\mathfrak{m}_X := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) | f|_X \equiv 0\}$. Thus

$$\mathrm{res} : \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X} \cong \mathcal{C}^\infty(X)$$

**Definition.** Taylor closed subsets of $\mathbb{R}^n$

Let $X \subseteq \mathbb{R}^n$ be a closed set. Consider the ideal

$$\mathfrak{m}_X^\infty := \left\{g \in \mathcal{C}^\infty(\mathbb{R}^n) \middle| \forall k \geq 0, \partial^k f|_X \equiv 0\right\}$$

and the $\mathbb{R}$-algebra

$$\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X^\infty} \to \mathcal{C}(X)$$

. We call $\left(X, \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X^\infty}\right)$ a **Taylor closed subset** of $\mathbb{R}^n$.

We have the quotient map

$$\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X^\infty} \to \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X} \cong \mathcal{C}^\infty(X)$$

thus a *morphism*

$$\iota : (X, \mathcal{C}^\infty(X)) \to \left(X, \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\mathfrak{m}_X^\infty}\right)$$

[1]

**Definition.** Definition

- *objects* are $(X, A)$ such that $X \subseteq \mathbb{R}^n$ is a closed subset and $A \to \mathcal{C}^\infty(X)$ is a $\mathbb{R}$-algebra morphism
- *morphisms* $f : (X_1, A_1) \to (X_2, A_2)$ are a pair
  - of continuous map $f : X_1 \to X_2$ such that $f^* : \mathcal{C}^\infty(X_2) \to \mathcal{C}^\infty(X_1)$
  - and a $\mathbb{R}$-algebra homomorphism $f^\# : A_2 \to A_1$
  - such that

$$\begin{tikzcd} A_1 \arrow[r] & \mathcal{C}^\infty(X_1) \\ A_2 \arrow[r] \arrow[u, "f^\#"] & \mathcal{C}^\infty(X_2) \arrow[u, "f^*"] \end{tikzcd}$$

    commutes

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Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Closed subsets of $\mathbb{R}^n$ equipped with smooth functions

And it has 39 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $_\Gamma\backslash^{\mathbb{R}^n}, {}_\Gamma\backslash^{\mathbb{C}^n}$
    - Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - Open subsets of $\mathbb{R}^n$ with $C^2$ boundary
    - Complex ODEs in $\mathbb{C}$
    - Circle packing on $\mathbb{R}^2$
    - Circle packing converges to the Riemann biholomorphism
    - Bounded connected open subsets of $\mathbb{C}^n$
    - Connected circular open subsets of $\mathbb{C}^n$
    - Continuous functions on $\mathbb{R}^d$
    - Dyadic cubes
    - Curves
    - Differentiable functions
    - Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner transform
    - Harmonic functions composed with conformal maps
    - Hilbert transform

- Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
- Holomorphic subsets of $\mathbb{C}^n$
- Regular hypersurfaces in $\mathbb{R}^{2n}$
- Orientable hypersurfaces in $\mathbb{R}^n$
- Laplace operator on $\mathbb{R}^n$
- $l^\infty(\mathbb{R})$
- Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- Local $\mathbb{R}$-algebras
- $l^p(\mathbb{R})$
- Metric density of subsets of $\mathbb{R}^n$
- Monotone functions on $\mathbb{R}$
- Cauchy integral of periodic functions
- Polygons with integer vertices
- Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- Discrete subgroups of $\mathbb{R}^n$
- Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- Ramified germs of smooth and holomorphic functions
- Open subsets of $\mathbb{R}^n$
- Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- Closed subsets of $\mathbb{R}^n$ equipped with smooth functions
- Quasi-analytic smooth functions on $\mathbb{R}$
- Star-shaped subsets of $\mathbb{R}^n$
- ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$

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**7.13. Definition.** A *smooth set* is a pair $(W, C^\infty(W))$, where $W$ is a closed subset $W \subset M$ of a smooth manifold $M$, and $C^\infty(W)$ is the algebra of smooth functions on $W$ defined as follows:

$$C^\infty(W) \stackrel{\text{def}}{=} \left\{ f\big|_W \mid f \in C^\infty(M) \right\}.$$

**Exercise.** Prove that theorems similar to 7.2 and 7.7 are valid for smooth sets as well.

**7.14 Exercise.** Describe the algebras $C^\infty(W)$ for the following cases:

1. $W = \mathbf{K}$ is the coordinate cross on the plane:

$$\mathbf{K} = \left\{ (x, y) \subset \mathbb{R}^2 \mid xy = 0 \right\}.$$

1. ↩