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Meromorphic functions on the complex plane

Definition. Meromorphic function on the complex plane

A holomorphic function

$$f : z_0 + D_R^* \rightarrow \mathbb{C}$$

is **meromorphic at** z_0 if

$$|f(z)| \xrightarrow{z \rightarrow z_0} \infty$$

which means

$$\forall M > 0 \exists r \in (0, R) : |z - z_0| < r \implies |f(z)| > M$$

f is holomorphic on $z_0 + RD^*$	type of (isolated) singularities	the Laurent series $f(z) = \sum_{n \in \mathbb{Z}} c_n (z - a)^n$	image of f on $z_0 + \mathbb{D}_R^*$
	removable singularity, holomorphic at z_0	$c_n = 0$ for all $n < 0$	bounded
$(z - z_0)^k f(z)$ has a <i>removable singularity</i> for some $k > 0$ but f does not	pole, <u>meromorphic at</u> z_0	eventually $c_n = 0$	$\ f(z)\ \rightarrow \infty$ as $\ z\ \rightarrow 0$
$(z - z_0)^k f(z)$ does not have a removable singularity for any $k \in \mathbb{N}$	essential singularity	c_n does not become 0 eventually	image is dense on $\mathbb{C}$ (actually reaches almost all values on $\mathbb{C}$)

- If f is meromorphic at z_0 then

$$\frac{1}{f} : z_0 + D_r \rightarrow \mathbb{C}$$

is a non-constant holomorphic function with $z_0 \mapsto 0$.

- This means by

- If $f : U \rightarrow \mathbb{C}$ is a non-constant holomorphic function with

$$f(z) \equiv \sum_{n \geq k} a_n (z - z_0)^n \quad \text{on } B(r, z_0)$$

for $z_0 \in U$ where k is the smallest positive integer such that $a_k \neq 0$.

- $\implies$

$$f(z) = (z - z_0)^k g(z) \quad \text{on } B(r_1, z_0) \subseteq B(r, z_0)$$

where g is a holomorphic function such that

$$g(z) :\equiv a_k + a_{k+1}(z - z_0) + \dots$$

- As $g(z_0) = a_k \neq 0$, this means there is a smaller disk on which g is non-zero

$$g \neq 0 \quad \text{on } B(r_2, z_0)$$

- $\implies f \neq 0$ on that same punched disk around z_0
- Thus **zeroes of non-constant holomorphic functions on an open subset of $\mathbb{C}$ are isolated in $\mathbb{C}$** , that is, the set of zeros $Z(f)$ is discrete in $\mathbb{C}$.

$$\frac{1}{f(z)} = (z - z_0)^k g(z)$$

where g is holomorphic and $g \neq 0$ on a small disk near z_0 and k is a positive integer.

- This mean $\frac{1}{g}$ is holomorphic on a small disk near z_0 and

$$f(z) = \frac{\frac{1}{g}(z)}{(z - z_0)^k}$$

- Thus for any function f that is meromorphic at z_0 , there is a holomorphic $h : z_0 + D_r \rightarrow \mathbb{C}$ and positive integer k called the **order of the pole at** z_0 such that

$$f(z) = \frac{h(z)}{(z - z_0)^k} \text{ on } z_0 + D_r^*$$

Moreover $h(z_0) \neq 0$.

Current note has 0 direct children and 0 total descendants.

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And it has 19 siblings.

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- subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
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 - Zeros and singularities of holomorphic functions